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Universidad Rafael Urdaneta

40 años de futuro 1973-2013

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Dirección: Universidad Rafael Urdaneta, Av. 2 c/Calle 86, entrada Sur del Parque Vereda del Lago, Maracaibo, Venezuela.
Teléf. (58) (261) - 2000887, Fax (58)(261) 2000868. Correo electrónico: revistatecnocientificauru@gmail.com

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Presentación

Esta edición especial de la Revista Tecnocientífica URU se presenta en ocasión del 76 aniversario del Dr. Shyam L. Kalla, habiendo dedicado más de cincuenta años al desarrollo de la investigación matemática. El Dr. Kalla es nacido en la India, pero es venezolano de corazón.

Durante los quince años que estuvo en Venezuela formó parte del personal docente y de investigación de la Universidad del Zulia, dejando como legado la formación de un grupo de investigadores que dio origen al Centro de Investigación de Matemática Aplicada, donde se han producido un gran numero de investigaciones enmarcadas en diversas áreas a fines.

El Dr. Kalla continuaría su labor en otros países, en los cuales ha dejado huellas de su talento y su gran virtud para la formación de investigadores.

Ha publicado trabajos en diversas áreas como Funciones Especiales, Ecuaciones Integrales, Transformadas Integrales, Cálculo Fraccional y Aproximación de Funciones, entre otras, en revistas científicas de reconocido prestigio.

Agradecemos a los autores y árbitros, tanto nacionales como internacionales su valiosa colaboración para concretar, esta Edición Especial en el área de Matemáticas, la cual presentamos a la comunidad científica

La Universidad Rafael Urdaneta consciente del aporte del Dr. Shyam L. Kalla al desarrollo de teorías y aplicaciones en Matemática, decidió constituir un Comité Organizador para producir esta Edición Especial en honor a tan ilustre profesor.

Prof. Susana Salinas
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Prof. Salvador Conde
Secretario-Académico
Universidad Rafael Urdaneta
Maracaibo, Zulia-Venezuela

Dear Professor Conde,

Thank you very much for publishing a special issue of the journal:

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Nº 6 Enero - Junio 2014

dedicated to me. This is great honor for me and I feel privileged and honored. Thanks for writing nice presentation for this special issue. Please convey my thanks and regards to members of the Editorial Board.

As a matter of fact, I enjoyed my stay in Maracaibo and I have had opportunity to work with very talented a dedicated researchers of LUZ and URU.

Best wishes and regards.

Shyam Kalla

Professor Shyam L. Kalla
Ph.D., D.Sc. F.N.A. (Argentina & Ukraine)
Department of Mathematics, VIHE

Dr. Shyam Lal Kalla

El Dr. Shyam Lal Kalla es un notable matemático nacido en la India, país en el cual realizó su educación y donde recibió los títulos de PhD en Matemáticas, en la Universidad de Rajasthan (1967) y Doctor en Ciencias Matemáticas de la Universidad de Jodhpur (1976).

Desde su graduación el Dr.Kalla se ha dedicado a la docencia y la investigación, campos en los cuales ha tenido una actuación extraordinaria. En este sentido, ha sido profesor e investigador en Matemáticas en las Instituciones siguientes: Departamento de Educación de Rajasthan (India), Instituto Nacional de Tecnología en Jaipan (India), Universidad de Jodhpur (India), Universidad Nacional de Tucumán (Argentina), División de Posgrado de la Facultad de Ingeniería de la Universidad del Zulia (Venezuela) y el Departamento de Matemáticas y Ciencias de la Computación de la Universidad de Kuwait. Actualmente se desempeña como profesor en el departamento de Matemáticas del Instituto Vyas de Educación Avanzada de Jodhpur (India).

En todas estas instituciones desarrolló amplias actividades de docencia e investigación, fomentando y creando grupos de investigadores que han alcanzado resonancia mundial.

Es autor o coautor de más de 400 trabajos en las áreas de Funciones Espaciales, Transformaciones Integrales, Ecuaciones Integrales, Calculo Fraccional, Transferencia de Calor, Problemas de Valores Límites, Polinomios Ortogonales, Ecuaciones Diferenciales, Expansiones Asintóticas, Teoría de Probabilidades y Modelaje Matemático, los cuales se han publicado en una gran variedad de revistas nacionales e internacionales.

Debido a este fecundo trabajo, ha sido merecedor de los siguientes premios y honores: Premio de Investigación Andrés Bello en LUZ, Venezuela 1983,1984 y 1994, Individuo de Numero de la Academia Nacional de Argentina desde 1984, Premio Francisco Eugenio Bustamante de la Universidad del Zulia, Venezuela en 1992 y Profesor Honorario en la División de Matemáticas del Instituto para la Ricerca di Base, en Monteroduni, Italia. Igualmente ha sido profesor visitante en diversas instituciones académicas de Europa, Asia y América Latina y miembro de varias instituciones académicas y profesionales.

Durante su estadía en Venezuela por 16 años en la Facultad de Ingeniería de la Universidad del Zulia, el Dr. Kalla desarrolló una labor encomiable como Profesor de Matemáticas en la División de Posgrado y se dedicó a formar y dirigir una apreciable cantidad de investigadores en diversas áreas de las Ciencias Matemáticas, los cuales hoy desarrollan sus trabajos en la Universidad del Zulia y en la Universidad Rafael Urdaneta. Pero, sobre todo, fue valiosísima su contribución al fortalecimiento de la Revista Técnica-Científica de la Facultad de Ingeniería y la creación del Centro de Investigaciones en Matemáticas Aplicadas (CIMA) que hoy, bajo la dirección de sus antiguos alumnos, continua realizando una labor encomiable en su campo

Sin lugar a dudas, el Dr. Kalla será recordado por su creación intelectual y su extraordinaria generosidad en su relación formadora y generadora de conocimientos con sus discípulos de América, Europa y Asia.

Profesor Salvador Conde
Secretario Académico
Universidad Rafael Urdaneta

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Sobre una generalización de la función hipergeométrica de Gauss

Nina Virchenko

Dedicated to Professor Shyam Kalla
On the occasion of his 76th birthday

Phys.-Math. Department of National Technical University of Ukraine
"KPI" – Kyiv – 03056, Ukraine.
nvirchenko@hotmail.com

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Resumen

El trabajo trata una nueva generación de la función hipergeométrica de Gauss, ${}_rF^{\tau,\beta}(a,b;c;z)$. Se demuestran las propiedades básicas de esta función (representación en serie, formulas diferenciales, relaciones fraccionales, representación integral, relación de tipo Erdelyi).

Palabras y frases claves: r-función hipergeométrica, función gamma (τ,β), – función hipergeométrica confluente.

On one generalization of the Gauss' hypergeometric function

Abstract

The paper devoted to the new generalization of the hypergeometric Gauss' function, ${}_rF^{\tau,\beta}(a,b;c;z)$. The basic properties of this function (the representation by series, the differential formulas, the fractional relations, integral representations, the relation of type Erdelyi) are proved.

2000 Mathematical Classification: 33C05, 33C20, 33B15.

Key words and phrases: r - hypergeometric function, gamma-function, (τ,β), – confluent hypergeometric function .

Introduction

Further studying of the special functions is prospective and very useful for the different branches of science.

The continuous development of the mechanics of solid medium, mathematical physics, probability theory, aerodynamics, astronomy and others has led to the generalization and creation of new classes of special functions [1], [2], [3].

It this article we consider the r – generalized Gauss' hypergeometric function, its properties.

Main results

Let us consider the r – generalized Gauss' hypergeometric function in the following form:

$$ {}_rF^{\tau,\beta}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \times \tag{1}$$

$$ \times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt, $$

where $\operatorname{Re}c > \operatorname{Re}b > 0, \{\tau,\beta\} \subset R, \tau > 0, \tau - \beta < 1; r > 0; r = 0, |z| < 1; \operatorname{Re}\gamma > \operatorname{Re}\alpha > 0,$ is the classical beta-function [5], ${}_1\Phi_1^{\tau,\beta}(...)$ is the (τ,β) generalized confluent hypergeometric function [4]:

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{1}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \; {}_1\Psi_1\left[\begin{matrix}(c;\tau) \\ (c;\beta)\end{matrix}\middle| zt^{\tau}\right]dt, \tag{2}$$

where ${}_1\Psi_1(...)$ is the Fox-Wright function [1]. As $\tau = \beta = 1, \alpha = \gamma$ in (1) we have:

$$ {}_r\tilde{F}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \; e^{-\frac{r}{t(1-t)}}dt. \tag{3}$$

As $\beta = \tau$, $r = 0$ in (1) we have the Gauss' hypergeometric function $F(a,b;c;z)$ [5].

Theorem 1 (on the representation of the function ${}_rF^{\tau,\beta}(a,b;c;z)$ by the series).

As the conditions:

$$ r \in \mathbf{C}, r > 0; r = 0, |z| < 1, z \in \mathbf{C}; \{a,b,c\} \subset \mathbf{C}, \operatorname{Re}c > \operatorname{Re}b > 0; \tag{4}$$
$$ \{\tau,\beta\} \subset R, \tau > 0, \tau - \beta < 1, \ \operatorname{Re}\gamma > \operatorname{Re}\alpha > 0 $$

the following formula for ${}_rF^{\tau,\beta}(a,b;c;z)$ is valid:

$$ {}_rF^{\tau,\beta}(a,b;c;z) = \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \; {}_{\tau,\beta}B_{\alpha}^{\gamma}(b+n, c-b; r) \frac{z^n}{n!}, \tag{5}$$

where $(a)_n$ is the Pochhammer' symbol, ${}_{\tau,\beta}B_{\alpha}^{\gamma}(...)$ is the (τ,β) - generalized beta-function [4]:

$$ {}_{\tau,\beta}B_{\alpha}^{\gamma}(x,y;r;\delta;\omega) = \int_0^1 t^{x-t}(1-t)^{y-1} \, {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t^{\delta}(1-t)^{\omega}}\right)dt, \tag{6}$$

here $\operatorname{Re}x > 0, \operatorname{Re}y > 0, \delta > 0, \omega > 0, {}_1\Phi_1^{\tau,\beta}(...)$ is the function (2). The series (5) converges absolutely as $|z| < 1$.

Proof. Using the function ${}_{\tau,\beta}B_{\alpha}^{\gamma}(x,y;r;\delta;\omega)$, its properties, the legality of interchanging the order of integration and summation, we have:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)\Gamma(a)}\sum_{n=0}^{\infty}\Gamma(a+n)\frac{z^n}{n!}\times $$

$$ \times\int_0^1 t^{b+n-1}(1-t)^{c-b-1}{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt= $$

$$ =\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\ {}_{\tau,\beta}B_{\alpha}^{\gamma}\left(b+n,c-b;r\right)\frac{z^n}{n!} $$

Let us prove absolute convergence of series (5) as $|z|<1$. The series (5) is the power series:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=A\sum_{n=0}^{\infty}c_n z^n, $$

where

$$ A=\frac{1}{B(b,c-b)\Gamma(a)},\qquad c_n=\frac{\Gamma(a+n)\,{}_{\tau,\beta}B_{\alpha}^{\gamma}(b+n,c-b;r)}{n!}\ . $$

Let us consider asymptotic behavior of c_n as $n\to\infty$. Using the formula for Γ – function [5]:

$$ \Gamma(z)=\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}[1+\mathrm{O}(z^{-1})],z\to\infty, \tag{7} $$

we receive:

$$ {}_{\tau,\beta}B_{\alpha}^{\gamma}\left(\tilde{a}+n,\tilde{b};r\right)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\frac{\Gamma(\tilde{b}-k)\Gamma(\tilde{a}-k+n)(-r)^k}{\Gamma(\tilde{a}+\tilde{b}-2k+n)k!}= $$

$$ =\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\frac{\Gamma(\tilde{b})\Gamma(1-\tilde{b})\sqrt{2\pi}(\tilde{a}+n-k)^{\tilde{a}-k+n-\frac{1}{2}}}{(-1)^k\Gamma(1-\tilde{b}+k)\sqrt{2\pi}(\tilde{a}+\tilde{b}-2k+n)^{\tilde{a}+\tilde{b}-2k+n-\frac{1}{2}}}\times $$

$$ \times\frac{e^{-\tilde{a}+k-n}}{e^{-\tilde{a}-\tilde{b}+2k-n}}\frac{(-r)^k}{k!}=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\times $$

$$ \times\frac{1}{\Gamma(1-\tilde{b}+k)}\frac{(nr)^k}{k!}=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{b}}G_{1,3}^{1,1}\left[\tau n\left|\begin{matrix}1-\alpha\\0,1-\gamma,\tilde{b}\end{matrix}\right.\right], $$

where $G_{p,q}^{m,n}[...]$ is the G-Mejer' function [1].

By the help of formula [1]:

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right]=G_{p,q}^{m,n}\left[\frac{1}{z}\left|\begin{matrix}1-b_1,\dots,1-b_q\\ 1-a_1,\dots,1-a_p\end{matrix}\right.\right]$$

we have as $n\to\infty$.

$${}_{\tau,\beta}B_{\alpha}^{\gamma}\left(\tilde{a}+n,\tilde{b};r\right)=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}G_{1,3}^{1,1}\left[\frac{1}{\tau n}\left|\begin{matrix}1,\gamma,1-\tilde{b}\\ \alpha\end{matrix}\right.\right]=$$

$$=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}\frac{\Gamma(1-\gamma)\Gamma(\tilde{b})}{\Gamma(1-\alpha)}=\frac{\Gamma(\gamma)\Gamma^2(\tilde{b})\Gamma(1-\tilde{b})\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)n^{\tilde{b}}}=\frac{C}{n^{\tilde{b}}},$$

where

$$C=\frac{\Gamma(\gamma)\Gamma^2(\tilde{b})\Gamma(1-\tilde{b})\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)}.$$

Consequently, as $\tilde{a}=b,\tilde{b}-c-b,n\to\infty,$ (7) we get:

$$c_n=\frac{\Gamma(a+n)C}{n!n^{c-b}}=\frac{C\sqrt{2\pi}(a+n)^{a+n-\frac{1}{2}}e^{-a-n}}{n!n^{c-b}}.$$

$$R=\lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|}=\lim_{n\to\infty}\left|\frac{c\sqrt{2\pi}(a+n+1)^{a+n+\frac{1}{2}}e^{-a-n-1}}{(n+1)!(n+1)c-b}\right|=1.$$

The series (5) converges absolutely as $|z|<1$.

Theorem 2. *Under the conditions of existence of* ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following formulas of differentiation are hold:*

$$\frac{d\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz}=\frac{ab}{c}\,{}_rF^{\tau,\beta}(a+1,b+1;c+1;z),\tag{8}$$

$$\frac{d^n\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz^n}=\frac{(a)_n(b)_n}{(c)_n}\,{}_rF^{\tau,\beta}(a+n,b+n;c+n;z),\tag{9}$$

$$z\frac{d\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz}=a\left({}_rF^{\tau,\beta}(a+1,b;c;z)-{}_rF^{\tau,\beta}(a,b;c;z)\right),\tag{10}$$

$$\frac{d^n}{dz^n}\left(z^{a+n-1}\,{}_rF^{\tau,\beta}(a,b;c;z)\right)=(a)_n z^{a-1}\,{}_rF^{\tau,\beta}(a+n,b;c;z).\tag{11}$$

Proof. Let us prove (9), (11). We have

$$\frac{d^n\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz^n}=\frac{d^n}{dz^n}\left(\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\times\right.$$

$$\left.\times{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt\right)=\frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}(1-zt)^{-a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)(-a)(-a-1)...(-a-n+1)(-1)^n dt =$$

$$=\frac{(a)_n(b)_n}{(c)_n}\,{}_rF^{\tau,\beta}(a+n,b+;c+n;z);$$

$$\frac{d^n}{dz^n}\left(z^{a+n+1}\,{}_rF^{\tau,\beta}(a,b;c;z)\right)=\frac{d^n}{dz^n}\left(\frac{1}{B(b,c-b)}\sum_{k=0}^{\infty}\Gamma(a+k)\frac{z^{a+n+k-1}}{k!}\times\right.$$

$$\times {}_{\tau,\beta}B_\alpha^\gamma(b+k,c-b;r)=\frac{1}{B(b,c-b)\Gamma(a)}\sum_{k=0}^{\infty}\Gamma(a+k)\,{}_{\tau,\beta}B_\alpha^\gamma(b+k,c-b;r)\times$$

$$\times\frac{z^{a+k-1}}{k!}(a+n+k-1)(a+n+k-2)...(a+k)=$$

$$=(a)_n z^{a-1}\,{}_rF^{\tau,\beta}(a+n,b;c;z).$$

Theorem 3. *For* $z,r\in\mathbf{C}, r>0; r=0, |z|<1; \{\tau,\beta\}\subset R_+,\ \tau-\beta<1,\ \{a,b,c\}\subset\mathbf{C}$, $\mathrm{Re}\gamma>\mathrm{Re}\alpha>0, \mathrm{Re}c>\mathrm{Re}b>0$ *the following functional relations are valid:*

$${}_rF^{\tau,\beta}(a+1,b;c;z)-{}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=\frac{b}{c}z\,{}_rF^{\tau,\beta}(a+1,b+1;c+1;z), \qquad (12)$$

$$b\ {}_rF^{\tau,\beta}(a,b+1;c;z)+(c-b-1)\,{}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=(c-1)\,{}_rF^{\tau,\beta}(a,b;c-1;z). \qquad (13)$$

Proof. Let us prove (13). Using (2):

$$b\ {}_rF^{\tau,\beta}(a,b+1;c;z)+(c-b-1)\,{}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=\frac{1}{B(b+1,c-b-1)}\int_0^1 t^b(1-t)^{c-b-2}(1-zt)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt+$$

$$=\frac{c-b-1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt=$$

$$=\frac{c}{B(b,c-b-1)}\int t^{b-1}(1-t)^{c-b-2}(1-zt)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt=$$

$$=(c-1)\,{}_rF^{\tau,\beta}(a,b;c-1;z).$$

Theorem 4. *In the case of fulfilling of the conditions of existence of* ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following integral representations for* ${}_rF^{\tau,\beta}(a,b;c;z)$ *are valid*:

$${}_rF^{\tau,\beta}(a,b;c;z)=\frac{2}{B(b,c-b)}\int_0^{\infty}\frac{(ch\omega)^{2a-2c+1}(sh\omega)^{2b-1}}{(ch^2\omega-zsh^2\omega)^a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{rch^4\omega}{sh^2\omega}\right)d\omega, \tag{14}$$

$$ {}_rF^{\tau,\beta}(a,b;c;z)=\frac{2^{b-a}}{B(b,c-b)}\int_0^\infty \frac{(ch\theta)^{2a-2c+1}(sh\theta)^{2c-a-b-1}}{\left(\frac{1}{2}-z+\frac{1}{2}ch\theta\right)^a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{rsh^4\theta}{2(ch\theta-1)^3}\right)d\theta, \tag{15}$$

$$ {}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^\infty e^{-bt}(1-e^{-t})^{c-b-1}(1-ze^{-t})^{-a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{re^t}{1-e^{-t}}\right)dt. \tag{16}$$

Proof of these relations can be straightforward make using in (1) the such changes of variable respectively:

$$t=\frac{sh^2\omega}{ch^2\omega}; \qquad t=\frac{2}{ch\theta+1}; \qquad t=e^{-t}.$$

Theorem 5. (*The relation of type Erdelyi for* ${}_rF^{\tau,\beta}(a,b;c;z)$)

As the conditions (4) for ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following relation is hold*:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=(1-z)^{-a}\,{}_rF^{\tau,\beta}\left(a,c-b;c;\frac{z}{z-1}\right). \tag{17}$$

Proof. Using (1) and the substitution

$t=1-v$

we receive:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^\infty (1-v)^{b-1}v^{c-b-1}(1-\frac{z}{z-1}v)^{-a}\times$$

$$\times(1-z)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{(1-v)v}\right)dv=$$

$$=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 (1-v)^{b-1}v^{c-b-1}(1-\frac{z}{z-1}v)^{-a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{(1-v)v}\right)dv=(1-z)^{-a}\quad {}_rF^{\tau,\beta}\left(a,c-b;c;\frac{z}{z-1}\right).$$

Here: $1-z(1-v)=(1-z)\left(1-\frac{z}{z-1}v\right)$.

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Representación integral de las $e^{\alpha x} x^k y^l$ – funciones de onda

Iryna Alexsandrovich y Nina Virchenko

National Technical University of Ukraine, Kyiu Ukrain

Dedicated to Prof. Shyam Kalla
On the occasion of his 76th Anniversary

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Resumen

El trabajo trata con una nueva generalización de funciones analíticas. Se deducen algunas representaciones integrales de las $e^{\alpha x} x^k y^l$ - funciones de onda ($k, l, \alpha - const, > 0$) y sus fórmulas de inversión. Como una aplicación de la teoría se formulan dos problemas y se resuelven estos en forma cerrada.

Palabras clave: Funciones analíticas, funciones de onda.

Integral representation of $e^{\alpha x} x^k y^l$ – wave functions

Abstract

The paper deals with a new generalization of analytical functions. Some integral representations of $e^{\alpha x} x^k y^l$ - wave functions ($k, l, \alpha - const, > 0$), their inversion formulas are derived. As an application of the theory two problems are formulated and solved in the closed form.

Key words: Analytical functions, wave functions.

Introduction

The generalized analytical functions of complex variables appear as a natural and rational generalization of analytical functions.

Picard [1], Beltrami [2], Vekya [3,4], Polozi [5], Manjavidze [6], and others have obtained many important results in the generalization of the theory of analytical functions of elliptic type and their applications. For example, Polozij [5] introduced the analytical functions, using the system:

$$\begin{cases} pu_x - qu_y - \upsilon_y = 0, \\ qu_x + pu_y + \upsilon_x = 0, \end{cases} \tag{1}$$

where p and q are the given real functions of x and y.

Later on the p – analytical and (p,q) – analytical functions found number of applications in different branches of the mathematics, mechanics etc… (axial symmetric theory of elasticity, in the theory of the filtration, solution of the boundary value problems of the theory of rotating covers).

In this paper we study the *p-wave functions* $f(z)=u+i\upsilon$ as the solutions of the following system of the *hyperbolic type*:

$$\begin{cases} pu_x = \upsilon_y, \\ pu_y = \upsilon_x, \end{cases} \tag{2}$$

where $p=e^{\alpha x}x^k y^l$, $(k,l,\alpha$ are positive constants). Some integral representations of p-wave functions and their inversion formulas are constructed.

The p-wave functions describe the processes of mechanics, hydromechanics, the supersonic stream of gas, are useful for solving of the boundary value problems of the mathematical physics etc… Let us notice, that the p-wave functions with the characteristic $p=x^k y^l$ are connected with Euler – Poisson – Darboux equation with two degenerate lines.

Integral representations of the $e^{\alpha x}x^k y^l$ -wave functions

Let G be the domain in the first quarter of the plane $z = x + iy$, bounded by the segments l_1 and l_2 of the real and imaginary axis, respectively, and some curves which are monotone with respect to x and y. The rectilinear segments which are drowned from arbitrary point of the domain orthogonally to axis x and y, belong to the domain G.

Now we state and prove the following theorems related to integral representations of the p-wave functions.

Theorem 1. If $\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)$ is the $e^{\alpha x}y^k$ -wave function [7] in the domain G and

$$\tilde{\varphi}_2(x,y)\big|_{l_2}=0, \tag{3}$$

then the function

$$\tilde{f}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)=\int_0^x\left[\tilde{\varphi}_1(\xi,y)x^{1-l}+i\tilde{\varphi}_2(\xi,y)\xi\right]\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi \tag{4}$$

will be $e^{\alpha x}y^k x^l$ -wave function $(k,l,\alpha$ are positive constants) in G, continuous to segment l_2 and satisfies the condition

$$\tilde{\upsilon}(x,y)\big|_{l_2}=0. \tag{5}$$

Proof. According to the conditions of the theorem the functions $\tilde{\varphi}_1(x,y)$ and $\tilde{\varphi}_2(x,y)$ satisfy to next system

$$\begin{aligned} e^{\alpha x}y^k\frac{\partial\tilde{\varphi}_1}{\partial x}&=\frac{\partial\tilde{\varphi}_2}{\partial y}, \\ e^{\alpha x}y^k\frac{\partial\tilde{\varphi}_1}{\partial y}&=\frac{\partial\tilde{\varphi}_2}{\partial x}. \end{aligned} \tag{6}$$

Let us show that for the function (4) the following relations are valid:

$$e^{\alpha x} y^k x^l \frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{\upsilon}}{\partial y}, \qquad e^{\alpha x} y^k x^l \frac{\partial \tilde{u}}{\partial y} = \frac{\partial \tilde{\upsilon}}{\partial x}. \tag{7}$$

In reality:

$$\tilde{\upsilon}_{\gamma} = \int_0^1 \tilde{\varphi}_{1y}(xt, y)(1-t^2)^{\frac{l}{2}-1} dt,$$

$$\tilde{\upsilon}_x = \frac{\partial}{\partial x}\int_0^1 \tilde{\varphi}_2(xt, y) x^l t (1-t^2)^{\frac{l}{2}-1} dt.$$

Then we have:

$$e^{\alpha x} y^k x^l \tilde{u}_y - \tilde{\upsilon}_x = \int_0^1 \left[e^{\alpha x} y^k x^l \tilde{\varphi}_{1y}(xt, y) - t^2 x^l \tilde{\varphi}_{2xt}(xt, y) \right] \times$$

$$\times \left(1-t^2\right)^{\frac{l}{2}-1} dt - l \int_0^1 x^{l-1} \tilde{\varphi}_2(xt, y) t (1-t^2)^{\frac{l}{2}-1} dt.$$

Taking into account condition $\tilde{\varphi}_2(x, y)\big|_{x=0} = 0$, we get:

$$e^{\alpha x} y^k x^l \tilde{u}_y - \tilde{\upsilon}_x = x^l \int_0^1 \left[e^{\alpha x} y^k \tilde{\varphi}_{1y}(xt, y) - \tilde{\varphi}_{2xt}(xt, y) \right] \left(1-t^2\right)^{\frac{l}{2}-1} dt. \tag{8}$$

Keeping in mind (6) we proved the second relation from (7). The validity of the first relation from (7) is proving analogously.

The validity of the condition (5) follows from the relation:

$$\tilde{\upsilon}(x, y) = \int_0^x \xi \tilde{\varphi}_2(\xi, y) \left(x^2 - \xi^2\right)^{\frac{l}{2}-1} d\xi = \int_0^1 x^l \tilde{\varphi}_2(xt, y) t \left(1-t^2\right)^{\frac{l}{2}-1} dt. \tag{9}$$

Theorem 2. If $\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{\upsilon}(x, y)$ is the $e^{\alpha x} y^k x^l$ - wave function with (5) in the domain G, then the function

$\tilde{\varphi}(z) = \tilde{\varphi}_1(x, y) + i\tilde{\varphi}_2(x, y)$ will be the $e^{\alpha x} y^k$ - wave function and have the following form:
$\tilde{\varphi}(z) = \tilde{\varphi}_1(x, y) + i\tilde{\varphi}_2(x, y)$

$$\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\dfrac{l}{2}\right)\Gamma\left(m-\dfrac{l}{2}+1\right)}\left\{\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\xi^{l-1}\tilde{u}(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}+\right.\\ \left.+i\dfrac{1}{x}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\tilde{\upsilon}(\xi,y)}{(d\xi^2)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}\right\},\quad l\neq 2n, m=\left[\dfrac{l}{2}\right],\\ \dfrac{2}{(n-1)!}\left\{x\dfrac{d^n}{(dx^2)^n}\left[x^{l-1}\tilde{u}(x,y)\right]+i\dfrac{d^n\tilde{\upsilon}(x,y)}{(dx^2)^n}\right\},\quad l=2n,\end{cases}\tag{10}$$

and the condition (3) is valid.

Proof. According to (4) $\tilde{\varphi}_1(x,y)$ and $\tilde{\varphi}_2(x,y)$ are the solutions of the following equations, respectively:

$$\tilde{u}(x,y)=x^{1-l}\int_0^x\tilde{\varphi}_1(\xi,y)\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi,\tag{11}$$

$$\tilde{\upsilon}(x,y)=\int_0^x\tilde{\varphi}_2(\xi,y)\xi\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi.\tag{12}$$

The equations (11), (12) are integral equations Abel' type. The solutions of these equations give (10).

Let us show that the function $\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)$ are the $e^{\alpha x}y^k$ - wave function.

Let us introduce the next designations:

$$\begin{aligned}h(x,y)&=e^{\alpha x}y^k\tilde{\varphi}_{1x}-\tilde{\varphi}_{2y},\\ M(x,y)&=e^{\alpha x}y^k\tilde{\varphi}_{1y}-\tilde{\varphi}_{2x},\\ \tilde{L}(x,y)&=e^{\alpha x}y^k x^l\tilde{u}_x-\tilde{\upsilon}_y\\ \tilde{M}(x,y)&=e^{\alpha x}y^k x^l\tilde{u}_y-\tilde{\upsilon}_x.\end{aligned}\tag{13}$$

Then we can rewrite (1) in the following form:

$$\tilde{M}(x,y)=x\int_0^x\tilde{M}(\xi,y)\xi\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi,\tag{14}$$

$$\tilde{L}(x,y)=\int_0^x h(\xi,y)\xi\left(x^2-\xi^2\right).\ d\xi.\tag{15}$$

The solutions of (14) and (15) have the following kind:

$$\tilde{M}(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\dfrac{l}{2}\right)\Gamma\left(m-\dfrac{l}{2}+1\right)}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\xi^{-1}M(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l\neq 2n, m=\left[\dfrac{l}{2}\right],\\ \dfrac{2x}{(n-1)!}\dfrac{d^n\left[x^{l-1}\tilde{M}(x,y)\right]}{(dx^2)^n},\quad l=2n;\end{cases}\tag{16}$$

$$h(x,y)=\begin{cases}\dfrac{2}{\Gamma\left(\dfrac{l}{2}\right)\Gamma\left(m-\dfrac{l}{2}+1\right)}\dfrac{1}{x}\dfrac{d}{dx}\displaystyle\int_0^x\dfrac{d^m\left[\tilde{L}(\xi,y)\right]}{\left(d\xi^2\right)^m}\dfrac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l\neq 2n, m=\left[\dfrac{l}{2}\right],\\ \dfrac{2x}{(n-1)!}\dfrac{d^n\tilde{L}(x,y)}{(dx^2)^n},\quad l=2n.\end{cases}\tag{17}$$

Because $\tilde{L}(x,y)=0$, $\tilde{M}(x,y)=0$, then, respectively, $L(x,y)=0$, $M(x,y)=0$, consequently, $\tilde{\varphi}(z)$ is the $e^{\alpha x}y^k -$ wave function. Let us show, that

$$\tilde{\varphi}_2(x,y)\big|_{l_2}=0\,.$$

i) Let $l = 2n$. Using (10) we have:

$$\frac{d^n}{(dx^2)^{n-1}}\left[x^{l-1}\tilde{u}(x,y)\right]=(n-1)!\int_0^x\tilde{\varphi}_1(\xi,y)d\xi,$$

$$\tilde{\varphi}_2(x,y)=\frac{1}{(n-1)!}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[\frac{1}{x}\frac{\partial\tilde{\upsilon}(x,y)}{\partial x}\right].$$

Since $\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)$ is $e^{\alpha x}y^kx^l -$ wave function, then we can rewrite the last relation in the form:

$$\tilde{\varphi}_2(x,y)=\frac{1}{(n-1)!}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[e^{\alpha x}y^kx^{l-1}\frac{\partial\tilde{u}}{\partial y}\right]=$$

$$=\frac{e^{\alpha x}y^k}{(n-1)!}\frac{\partial}{\partial y}\frac{d^{n-1}}{(dx^2)^{n-1}}\left[x^{l-1}\tilde{u}\right]$$

or:

$$\tilde{\varphi}_2(x,y)=e^{\alpha x}y^k\int_0^x\frac{\partial\tilde{\varphi}_1(\xi,y)}{\partial y}d\xi,$$

from here

$$\tilde{\varphi}_2(x,y)\big|_{x=0}=0\,\cdot$$

ii) Let $l \neq 2n$. From (11) we get:

$$\frac{d^{m-1}}{(dx^2)^{m-1}}\left[x^{l-1}\tilde{u}(x,y)\right] = \int_0^x \tilde{\varphi}_1(\xi,y)\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)\left(x^2-\xi^2\right)^{\frac{l}{2}-m} d\xi \cdot$$

According to (10) we have:

$$\tilde{\varphi}_2(x,y) = \frac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\frac{1}{x}\frac{d}{dx}\int_0^x \frac{d^m\tilde{\upsilon}(\xi,y)}{\left(d\xi^2\right)^m}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}},\ l \neq 2n, m = \left[\frac{l}{2}\right].$$

Let us consider

$$\frac{d^m\tilde{\upsilon}(\xi,y)}{\left(d\xi^2\right)^m} = \frac{d^{m-1}}{(d\xi^2)^{m-1}}\left[\frac{1}{2\xi}\frac{\partial\tilde{\upsilon}}{\partial\xi}\right] = \frac{d^{m-1}}{(d\xi^2)^{m-1}}\left[\frac{1}{2\xi}e^{\alpha\xi}y^k\xi^l\tilde{u}_y\right] =$$

$$= \frac{1}{2}y^k e^{\alpha\xi}\frac{\partial}{\partial y}\left[\frac{d^{m-1}(\xi^{l-1}\tilde{u})}{\left(d\xi^2\right)^{m-1}}\right] = \frac{1}{2}y^k e^{\alpha\xi}\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)\times$$

$$\times\int_0^\xi \frac{\partial\tilde{\varphi}_1(\tau,y)}{\partial y}\left(\xi^2-\tau^2\right)^{\frac{l}{2}-m} d\tau.$$

Now we transform $\tilde{\varphi}_2(x,y)$:

$$\tilde{\varphi}_2(x,y) = \frac{y^k\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}\frac{1}{x}\frac{d}{dx}\int_0^x \frac{e^{\alpha\xi}\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-m}}\int_0^\xi \frac{\partial\tilde{\varphi}_1(\tau,y)}{\partial y}\left(\xi^2-\tau^2\right)^{\frac{l}{2}-m} d\tau.$$

Letting $\tau = \xi t, \xi = x\eta$ we get:

$$\tilde{\varphi}_2(x,y) = \frac{\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}y^k\left(3x\int_0^1 e^{\alpha x\eta}\eta^{l-2m+2}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\right)\times$$

$$\times\int_0^1 \frac{\partial\tilde{\varphi}_1(x\eta t,y)}{\partial y}\left(1-t^2\right)^{\frac{l}{2}-m} dt + x^2\int_0^1 \alpha e^{\alpha x\eta}\eta^{l-2m+3}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\int_0^1 \frac{\partial\tilde{\varphi}_1(x\eta t,y)}{\partial y}\left(1-t^2\right)^{\frac{l}{2}-m} dt\Bigg) +$$

$$+\frac{\left(\frac{l}{2}-1\right)\left(\frac{l}{2}-2\right)...\left(\frac{l}{2}-m+1\right)}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)}x^2\left(\int_0^1 e^{x\alpha\eta}\eta^{l-2m+3}\left(1-\eta^2\right)^{m-\frac{l}{2}} d\eta\int_0^1\left(-k\frac{1}{y}\frac{\partial\tilde{\varphi}_2(x\eta t,y)}{\partial y}+\right.\right.$$

$$+\frac{\partial^2\tilde{\varphi}_2(x\eta t,y)}{\partial y^2}\Bigg)e^{-\alpha x\eta t}t\left(1-t^2\right)^{\frac{l}{2}-m}dt\Bigg).$$

Hence, $\tilde{\varphi}_2(x,y)\big|_{x=0}=0$. The proof of the theorem is complete.

Definition. The function $u_0(x,y)$ will be called the real wave function in the domain G, if $u_0\in C^2(G)$ and holds the equation:

$$\frac{\partial^2 u_0}{\partial x^2}-\frac{\partial^2 u_0}{\partial y^2}=0 \tag{18}$$

Let us remark the following connection between $e^{\alpha x}y^k-$ wave functions and the wave function [3].

If $u_0(x,y)$ is an arbitrary real wave function in G with the condition

$$\left.\frac{\partial u_0}{\partial y^2}\right|_{y=0}=0 \tag{19}$$

then the integral representation of $e^{\alpha x}y^k-$ wave functions has the next form:

$$\tilde{\varphi}(z)=\tilde{\varphi}_1(x,y)+i\tilde{\varphi}_2(x,y)=y^{1-k}e^{-\frac{\alpha x}{2}}\int_0^y u_0(x,\tau)\left(y^2-\tau^2\right)^{\frac{k}{2}-1}{}_0F_1\left(\frac{k}{2};-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)d\tau+$$

$$+\frac{i}{k}e^{\frac{\alpha x}{2}}\int_0^y\left[\frac{\partial u_0(x,\tau)}{\partial x}-\frac{\alpha}{2}u_0(x,\tau)\right]{}_0F_1\left(\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{\frac{k}{2}}d\tau, \tag{20}$$

where ${}_0F_1(\nu;z)$ is the partial case of the generalized hypergeometric function [8]

$${}_0F_1(\nu;z)=\Gamma(\nu)\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+\nu)n!}.$$

The solution of the equation (20) with respect to $u_0(x,y)$ has the following form [9]:

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{2e^{\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)}\times \tag{21}$$

$$\times\frac{d}{dy}\int_0^y\frac{d^m\left[\tau^{k-1}\tilde{\varphi}_1(x,\tau)\right]}{(d\tau^2)^m}{}_0F_1\left(m-\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{m-\frac{k}{2}}\tau d\tau+$$

$$+i\cdot 2\frac{e^{-\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)}\frac{\partial}{\partial y}\left\{\frac{1}{y}\frac{\partial}{\partial y}\int_0^y\frac{d^m\tilde{\varphi}_2(x,\tau)}{(d\tau^2)^m}{}_0F_1\left(m-\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\times\right.$$

$$\left.\times\frac{\tau d\tau}{\left(y^2-\tau^2\right)^{\frac{k}{2}-m}}\right\},\quad k\neq 2n,\ m=\left[\frac{k}{2}\right];$$

as $k = 2n$

$$u_0(x,y) + i\left[\frac{\partial u_0(x,y)}{\partial x} - \frac{\alpha}{2}u_0(x,y)\right] = \frac{2e^{\frac{\alpha}{2}x}}{(n-1)!}y\frac{d^n\left[y^{k-1}\tilde{\varphi}_{1n}(x,y)\right]}{(dy^2)^m} + \tag{22}$$

$$+e^{\frac{\alpha}{2}x}\frac{\alpha y}{(n-1)!}\int_0^y \frac{d^n\left[\tau^{k-1}\tilde{\varphi}_1(x,\tau)\right]}{(d\tau^2)^n}\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)\frac{\tau d\tau}{\sqrt{y^2-\tau^2}} +$$

$$+i\frac{2e^{-\frac{\alpha}{2}x}}{(n-1)!}\frac{\partial}{\partial y}\left\{\frac{d^n\tilde{\varphi}_2(x,y)}{(dy^2)^n} + \frac{\alpha}{2}\int_0^y \frac{d^n\tilde{\varphi}_2(x,\tau)}{(d\tau^2)^n}\frac{\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau\right\}.$$

Let us construct integral representations of the $e^{\alpha x}y^k x^l$ – wave functions by means of real wave functions.

After some transformations we can write (4) in the following form:

$$\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{\upsilon}(x,y) = x^{1-l}y^{1-k}\int_0^x e^{-\frac{\alpha\xi}{2}}(x^2-\xi^2)^{\frac{l}{2}-1}d\xi\int_0^y u_0(\xi,\tau)\,{}_0F_1\left(\frac{k}{2};-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\times$$

$$\times\left(y^2-\tau^2\right)^{\frac{k}{2}-1}d\tau + \tag{23}$$

$$+\frac{i}{k}\int_0^x \xi e^{\frac{\alpha\xi}{2}}\left(x^2-\xi^2\right)^{\frac{l}{2}-1}d\xi\int_0^y\left[\frac{\partial u_0(\xi,\tau)}{\partial\xi} - \frac{\alpha}{2}u_0(\xi,\tau)\right]{}_0F_1\left(\frac{k}{2}+1;-\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{\frac{k}{2}}d\tau.$$

In order that find the inversion formula of (23) we use (21), (10). We obtain:

$$u_0(x,y) + i\left[\frac{\partial u_0(x,y)}{\partial x} - \frac{\alpha}{2}u_0(x,y)\right] = \frac{4e^{\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\times$$

$$\times\frac{d}{dy}\int_0^y \tau\left(y^2-\tau^2\right)^{m-\frac{k}{2}}\frac{d^m}{(d\tau^2)^m}\left[\tau^{k-1}\frac{d}{dx}\int_0^x \frac{d^r\left[\xi^{l-1}\tilde{u}(\xi,\tau)\right]}{(d\xi^2)^r}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-r}}\right]\times$$

$$\times{}_0F_1\left(m-\frac{k}{2}+1;\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)d\tau + \tag{24}$$

$$+i\frac{4e^{-\frac{\alpha}{2}x}}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\frac{\partial}{\partial y}\left\{\frac{1}{y}\frac{\partial}{\partial y}\int_0^y \frac{d^m}{(d\tau^2)^m}\left[\frac{1}{x}\frac{d}{dx}\int_0^x \frac{d^r\tilde{\upsilon}(\xi,\tau)}{(d\xi^2)^r}\frac{\xi d\xi}{\left(x^2-\xi^2\right)^{\frac{l}{2}-r}}\right]\times\right.$$

$$\times {}_0F_1\left(m-\frac{k}{2}+1;\frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\frac{\tau d\tau}{\left(y^2-\tau^2\right)^{\frac{k}{2}-m}}\Bigg\},$$

$$\left(k\neq 2n, l\neq 2r, m=\left[\frac{k}{2}\right], r=\left[\frac{l}{2}\right]\right);$$

$$u_0(x,y)+i\left[\frac{\partial u_0(x,y)}{\partial x}-\frac{\alpha}{2}u_0(x,y)\right]=\frac{4e^{\frac{\alpha}{2}x}}{(n-1)!(p-1)!}y\frac{d^n}{(dy^2)^n}\left[y^{k-1}x\frac{d^p}{(dx^2)^p}\left(x^{l-1}\tilde{u}(x,y)\right)\right]+$$

$$+\frac{2\alpha xye^{\frac{\alpha}{2}x}}{(n-1)!(p-1)!}\int_0^y\frac{d^n}{(dx^2)^n}\left[\tau^{k-1}\frac{d^p}{(dx^2)^p}\left(x^{l-1}\tilde{u}(x,\tau)\right)\right]\frac{\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau+ \tag{25}$$

$$+i\frac{4e^{-\frac{\alpha}{2}x}}{(n-1)!(p-1)!}\frac{\partial}{\partial y}\left\{\frac{d}{(dy^2)^n}\left(\frac{d^p\tilde{\upsilon}(x,y)}{(dx^2)^p}\right)+\frac{\alpha}{2}\int_0^y\frac{d^n}{(dx^2)^n}\left(\frac{d^p\tilde{\upsilon}(x,y)}{(dx^2)^p}\right)\frac{\mathrm{I}_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}}\tau d\tau\right\},$$

$\left(k=2n, l=2p, n, p \text{ are integer}\right).$

The formulas (23), (24) give possibility to reduce the boundary value problems in the class of the $e^{\alpha x}y^kx^l$ – wave functions to the corresponding boundary value problems for the homogeneous wave equation. Let us consider some problems.

In the domain $D=\{(x,y):0<x<\infty, 0<y<\infty\}$ find the $e^{\alpha x}y^kx^l$ – wave functions $\tilde{f}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)$, which satisfy the following conditions:

$$\tilde{u}(x,y)\big|_{y=0}=\varphi(x),\quad 0<x<\infty, \tag{26}$$

$$\tilde{f}(z)\big|_{x=0}=\Phi(y),\quad 0<y<\infty, \tag{27}$$

where the functions $\varphi(x),\Phi(y)$ are the given continuously differentiable functions.

The solution of this problem we find using (23). For $u_0(x,y)$ we get next boundary conditions:

$$u_0(x,y)=\varphi_0(x)=\begin{cases}\dfrac{4\Gamma\left(\frac{k+1}{2}\right)e^{\frac{\alpha x}{2}}}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(r-\frac{l}{2}+1\right)}\dfrac{d}{dx}\displaystyle\int_0^x\frac{d^r\left[\xi^{l-1}\varphi(\xi)\right]}{(d\xi^2)^r}\frac{\xi d\xi}{(x^2-\xi^2)^{\frac{l}{2}-r}}, l\neq 2n, r=\left[\frac{l}{2}\right],\\ \dfrac{4\Gamma\left(\frac{k+1}{2}\right)e^{\frac{\alpha x}{2}}}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)(n-1)!}x\dfrac{d^r\left[x^{l-1}\varphi(x)\right]}{(dx^2)^n}, l=2n.\end{cases} \tag{28}$$

$$\frac{\partial u_0(x,0)}{\partial y} = 0, \qquad 0 < x < \infty; \tag{29}$$

$$u_0(0,y) = \Phi_0(y) = \begin{cases} \dfrac{4\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}+1\right)} \dfrac{d}{dy}\displaystyle\int_0^y \dfrac{d^m\left[\tau^{k-1}\Phi(\tau)\right]}{(d\tau^2)^m} {}_0F_1\left(m-\frac{k}{2}+1; \frac{\alpha^2}{16}\left(y^2-\tau^2\right)\right)\left(y^2-\tau^2\right)^{m-\frac{k}{2}} \tau d\tau, k \neq 2n, m = \left[\frac{k}{2}\right], \\ \dfrac{4\Gamma\left(\frac{l+1}{2}\right)e^{\frac{\alpha x}{2}}}{(n-1)!\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)} y \dfrac{d^n\left[y^{k-1}\Phi(y)\right]}{(dy^2)^n} + \dfrac{2\alpha\Gamma\left(\frac{l+1}{2}\right)}{(n-1)!\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)} y\displaystyle\int_0^y \dfrac{d^n\left[\tau^{k-1}\Phi(\tau)\right]}{(d\tau^2)^n} \times \\ \times \dfrac{I_1\left(\frac{\alpha}{2}\sqrt{y^2-\tau^2}\right)}{\sqrt{y^2-\tau^2}} \tau d\tau, \qquad k = 2n. \end{cases} \tag{30}$$

Using (28) - (30), d'Alembert formula, and formula (23) we obtain the unknown solution.

In the domain $D = \{(x,y): 0 < x < q, y > 0\}$ find the $e^{\alpha x} y^k x^l$ – wave functions $\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{\upsilon}(x,y)$, which satisfy the following conditions:

$$\tilde{u}(x,y)\big|_{y=0} = \varphi(x), \qquad 0 < x < q,$$

$$\tilde{f}(z)\big|_{x=0} = \Phi_1(y), \quad \tilde{f}(z)\big|_{x=q} = \Phi_2(y) \quad 0 < y < \infty, \tag{31}$$

where the functions $\varphi(x), \Phi_1(y), \Phi_2(y)$ are the given continuously differentiable functions.

The solution of this problem we find using (23). For real wave function $u_0(x,y)$ we receive the following boundary conditions:

$$u_0(x,0) = \varphi_0(x)$$
$$\frac{\partial u_0(x,0)}{\partial y} = 0, \qquad 0 < x < q; \tag{32}$$

$$u_0(0,y) = \Phi_1^0(y), \quad u_0(q,y) = \Phi_2^0(y), \quad 0 < y < \infty.$$

Using (30) we get $\Phi_1^0(y)$, $\Phi_2^0(y)$.

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Algunos resultados sobre la función de Bessel de tres variables

Ana Isolina Prieto[1], Josefina Matera[1], Leda Galué[1] y Susana Salinas[1,2]

[1]Universidad del Zulia. Facultad de Ingeniería. Centro de Investigación de Matemática Aplicada (CIMA). Apartado de correo 10482.
aisolinap@hotmail.com, pinamatera@yahoo.es, ledagalue@hotmail.com

[2]Departamento de Ciencias Básicas. Facultad de Ingeniería. Universidad Rafael Urdaneta.
susanaderomero@hotmail.com
Maracaibo-Venezuela

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Resumen

Las funciones especiales han jugado un rol importante en el desarrollo de la Matemática pura y las teorías físicas, estas funciones son importantes para científicos e ingenieros en muchas áreas de aplicación. En particular, las funciones de Bessel son de gran interés en el compo de las funciones espaciales. Las funciones de Bessel están relacionadas con la teoría de potencial, procesos multifotón y en campos de radiación y, en general, en problemas de física, matemática e ingeniería. En este trabajo se introduce la definición de la función de Bessel de tres variables, dos parámetros y un índice; además, se presentan propiedades de simetría, relaciones de recurrencia y ecuaciones diferenciales que satisfacen estas funciones.

Palabras clave: Función de Bessel de tres variables, relaciones de recurrencia, ecuaciones diferenciales.

Some results on Bessel function of three variables

Abstract

Special functions have played an important roll in the development of pure mathematics and theorical physics, these functions are important to scientists and engineers in many areas of application. In particular Bessel functions are great interest in the field of special functions. The Bessel functions are related with potential theory, multifoton processes, radiation fields and generally physics, mathematics and engineering problems. In this paper the definition of the Bessel function of three variables, two parameters and one index are introduced, moreover symmetry properties, recurrence relationships and differential equations that satisfy are presented.

Key words: Bessel functions of three variables, recurrence relationships, differential equation.

Introducción

Las funciones especiales son igualmente importantes para las matemáticas pura y aplicada [1]. Un tipo importante de estas funciones son las funciones de Bessel. El análisis de las funciones de Bessel ha abierto nuevos y fascinantes escenarios debido a que permite resolver una amplia variedad de problemas, por ejemplo, aplicación en la solución de ecuaciones diferenciales en matemática, en procesos multifoton [2], en el campo de radiación [3] y en otras ramas de la ciencia y tecnología.

Muchos autores han definido y estudiado diferentes formas generalizadas de la función de Bessel: C. Chiccoli et al. [4], proveen una visión unificada de la teoría de las funciones de Bessel, L. Galué [5] introduce las series Kapteyn para las funciones de Bessel, G. Dattoli et al. [6] introduce un método para derivar familias de funciones generadoras de las funciones de Bessel, Pathan et al. [7] obtienen diferentes funciones generadoras para la función de Bessel generalizada de dos variables y un parámetro $J_n(x, y; \tau)$ definida por

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\sum_{n=-\infty}^{+\infty} t^n J_n(x, y; \tau). \tag{1}$$

donde x, y son variables reales y t, τ son parámetros complejos diferentes de cero con $|t|, |\tau| < \infty$.

Para $y = 0$, la función (1) se reduce a la conocida función generadora de una variable de la función cilíndrica de Bessel $J_n(x)$, esto es

$$exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty} t^n J_n(x). \tag{2}$$

Consecuentemente, esto ha inspirado la investigación de la función de Bessel de tres variables, dos parámetros y un índice; aquí se introduce la función generadora, propiedades de simetría, relaciones de recurrencia y ecuaciones diferenciales que satisfacen.

Definición

Se introduce la función de Bessel generalizada de tres variables, dos parámetros y un índice $J_n(x, y, z; \tau, \delta)$ mediante la función generadora

$$exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]=\sum_{n=-\infty}^{\infty} t^n J_n(x, y, z; \tau, \delta). \tag{3}$$

donde x, y y z son variables reales y t, τ y δ son parámetros complejos diferentes de cero con $|t|, |\tau|$ y $|\delta| < \infty$.

Si en (3) se hace $z = 0$, se obtiene la función de Bessel (1) dada por Pathan [7].

Es obvio que si $y = z = 0$, (3) reduce a la conocida función de Bessel $J_n(x)$ dada en (2).

Demostración

Aplicando las propiedades de la exponencial del lado izquierdo de (3)

$$exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]=$$

$$exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]exp\left[\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]exp\left[\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]$$

usando (2)

$$=\sum_{m=-\infty}^{\infty} t^m J_m(x) \sum_{h=-\infty}^{\infty} (t^2\tau)^h J_h(y) \sum_{k=-\infty}^{\infty} (t^3\delta)^k J_k(z)$$

$$=\sum_{m,h,k=-\infty}^{\infty} t^{m+2h+3k}\, \tau^h \delta^k J_m(x)\, J_h(y)\, J_k(z).$$

Si $n = m + 2h + 3k, \Rightarrow m = n - 2h - 3k$

$$=\sum_{n=-\infty}^{\infty} t^n \sum_{h,k=-\infty}^{\infty} \tau^h \delta^k J_{n-2h-3k}(x)\, J_h(y)\, J_k(z).$$

Comparando con (3), resulta

$$J_n(x, y, z; \tau, \delta) = \sum_{h,k=-\infty}^{\infty} \tau^h \delta^k J_{n-2h-3k}(x)\, J_h(y)\, J_k(z). \tag{4}$$

representación en serie de $J_n(x, y, z; \tau, \delta)$.

Propiedades de simetría de $J_n(x, y, z; \tau, \delta)$

Las propiedades de simetría de la función de Bessel $J_n(x, y, z; \tau, \delta)$ pueden inferirse de la serie (4) y de la definición de la función de Bessel de primera clase de orden n [1,pág. 99 No. (5.2.2)].

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k!\,(n+k)!}, \quad |z| < \infty. \tag{5}$$

la cual satisface las propiedades de simetría

$$J_{-n}(x) = (-1)^n J_n(x) = J_n(-x) \tag{6}$$

A continuación se presentan algunas propiedades de simetría para $J_n(x, y, z; \tau, \delta)$

$$J_{-n}\left(x, y, z; \frac{1}{\tau}, \frac{1}{\delta}\right) = (-1)^n J_n(x, y, z; -\tau, \delta) = J_n(-x, -y, -z; \tau, \delta) \tag{7}$$

$$J_{-n}(x, y, z; \tau, \delta) = (-1)^n J_n(x, y, z; \tau, -\delta) = J_n(-x, -y, z; -\tau, \delta) \tag{8}$$

$$J_{-n}(x, y, z; \frac{1}{\tau}, \frac{-1}{\delta}) = (-1)^n J_n(x, y, z; -\tau, -\delta) = J_n(-x, y, z; -\tau, \delta) \tag{9}$$

$$J_{-n}\left(x, y, z; \frac{-1}{\tau}, \frac{1}{\delta}\right) = (-1)^n J_n(x, y, z; \tau, \delta) = J_n(-x, y, z; \tau, -\delta) \tag{10}$$

Se pueden calcular muchas otras relaciones de recurrencia para la función $J_n(x, y, z; \tau, \delta)$. A continuación se presenta la demostración de una de ellas y de manera similar, se pueden demostrar las demás propiedades de simetría.

Demostración de (7)

Sustituyendo x por $-x$, y por $-y$,z por $-z$ en (41), se tiene

$$J_n(-x, -y, -z; \tau, \delta) = \sum_{h,k=-\infty}^{\infty} \tau^h \delta^k J_{n-2h-3k}(-x) J_h(-y) J_k(-z).$$

usando (6)

$$\sum_{h,k=-\infty}^{\infty} \tau^h \delta^k (-1)^{n-2h-3k} J_{n-2h-3k}(x)(-1)^h J_h(y)(-1)^k J_k(z)$$

$$= (-1)^n \sum_{h,k=-\infty}^{\infty} (-\tau)^h \delta^k J_{n-2h-3k}(x) J_h(y) J_k(z)$$

obteniendo así el resultado intermedio de (7),

$$J_n(-x, -y, -z; \tau, \delta) = (-1)^n J_n(x, y, z; -\tau, \delta)$$

De (4) y usando (6)

$$J_n(-x, -y, -z; \tau, \delta) = \sum_{h,k=-\infty}^{\infty} \tau^h \delta^k J_{n-2h-3k}(-x)_h(-y) J_k(-z)$$

$$= \sum_{h,k=-\infty}^{\infty} \tau^h \delta^k J_{-n+2h+3k}(x) J_{-h}(y) J_{-k}(z)$$

con $h=-s$ y $k=-r$, se tiene

$$= \sum_{s,r=-\infty}^{\infty} (\tau)^{-s}(\delta)^{-r} J_{-n-2s-3r}(x) J_s(y)\, J_r(z)$$

$$J_n(-x,-y,-z;\tau,\delta) = J_{-n}\left(x, y, z; \frac{1}{\tau}, \frac{1}{\delta}\right).$$

Demostrando así la propiedad de simetría (7).

Relaciones de recurrencia

Las relaciones de recurrencia para $J_n(x, y, z; \tau, \delta)$ son:

$$\frac{\partial}{\partial x} J_n(x,y,z;\tau,\delta) = \frac{1}{2}[J_{n-1}(x,y,z;\tau,\delta) - J_{n+1}(x,y,z;\tau,\delta)] \tag{11}$$

$$\frac{\partial}{\partial y} J_n(x,y,z;\tau,\delta) = \frac{1}{2}\left[\tau J_{n-2}(x,y,z;\tau,\delta) - \frac{1}{\tau} J_{n+2}(x,y,z;\tau,\delta)\right] \tag{12}$$

$$\frac{\partial}{\partial z} J_n(x,y,z;\tau,\delta) = \frac{1}{2}\left[\delta J_{n-3}(x,y,z;\tau,\delta) - \frac{1}{\delta} J_{n+3}(x,y,z;\tau,\delta)\right] \tag{13}$$

$$\frac{\partial}{\partial \tau} J_n(x,y,z;\tau,\delta) = \frac{y}{2}\left[J_{n-2}(x,y,z;\tau,\delta) + \frac{1}{\tau^2} J_{n+2}(x,y,z;\tau,\delta)\right] \tag{14}$$

$$\frac{\partial}{\partial \delta} J_n(x,y,z;\tau,\delta) = \frac{z}{2}\left[J_{n-3}(x,y,z;\tau,\delta) + \frac{1}{\delta^2} J_{n+3}(x,y,z;\tau,\delta)\right] \tag{15}$$

$$n\, J_n(x,y,z;\tau,\delta) = \frac{x}{2}\,[J_{n-1}(x,y,z;\tau,\delta) + J_{n+1}(x,y,z;\tau,\delta)]$$

$$+\, y\left[\tau J_{n-2}(x,y,z;\tau,\delta) + \frac{1}{\tau}\, J_{n+2}(x,y,z;\tau,\delta)\right]$$

$$+\, \frac{3z}{2}\left[\delta J_{n-3}(x,y,z;\tau,\delta) + \frac{1}{\delta} J_{n+3}(x,y,z;\tau,\delta)\right] \tag{16}$$

Demostración de (11)

Sea w la función generadora dada en (3),

$$w = exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x, y, z; \tau, \delta). \tag{17}$$

Derivando w con respecto de x

$$\frac{\partial w}{\partial x} = exp\left[\frac{x}{2}\left(1-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]\cdot\left[\frac{1}{2}\left(t-\frac{1}{t}\right)\right]$$

esto es,

$$\frac{\partial w}{\partial x} = \frac{w}{2}\left(t-\frac{1}{t}\right) = \frac{wt}{2} - \frac{w}{2t}$$

Sustituyendo w por su desarrollo en serie (17)

$$\frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty} t^n J_n(x, y, z; \tau, \delta)$$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n+1} J_n(x, y, z; \tau, \delta) - \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n-1} J_n(x, y, z; \tau, \delta)$$

realizando cambio de índices

$$\sum_{n=-\infty}^{\infty} t^n \frac{\partial}{\partial x} J_n(x, y, z; \tau, \delta)$$
$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} t^n J_{n-1}(x, y, z; \tau, \delta) - \sum_{n=-\infty}^{\infty} t^n J_{n+1}(x, y, z; \tau, \delta)\right]$$

Comparando se obtiene la relación de recurrencia (11).

De manera similar se deriva w dada en (17) con respecto de las variables y y z y se obtienen, respectivamente, las relaciones (12) y (13).

Demostración de (14)

Derivando parcialmente$w$ con respecto de τ,

$$\frac{\partial w}{\partial \tau} = exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]\frac{y}{2}\left(t^2+\frac{1}{t^2\tau^2}\right)$$

$$= w\left[\frac{y}{2}\left(t^2+\frac{1}{t^2\tau^2}\right)\right] = \frac{wyt^2}{2} + \frac{wy}{2t^2\tau^2}$$

usando el desarrollo en serie de w dado en (17)

$$\sum_{n=-\infty}^{\infty} t^n J_n(x,y,z;\tau,\delta) = \frac{y}{2}\left[\sum_{n=-\infty}^{\infty} t^{n+2} J_n(x,y,z;\tau,\delta) + \frac{1}{\tau^2}\sum_{n=-\infty}^{\infty} t^{n-2} J_n(x,y,z;\tau,\delta)\right]$$

realizando la derivada de lado izquierdo y efectuando cambio de índices

$$\sum_{n=-\infty}^{\infty} t^n \frac{\partial}{\partial\tau} J_n(x,y,z;\tau,\delta) = \frac{y}{2}\left[\sum_{n=-\infty}^{\infty} t^{n} J_{n-2}(x,y,z;\tau,\delta) + \frac{1}{\tau^2}\sum_{n=-\infty}^{\infty} t^{n} J_{n+2}(x,y,z;\tau,\delta)\right]$$

se llega a demostrar la relación (14). Análogamente, de (17) derivando a w con respecto de δ, se obtiene (15).

Demostración de (16)

Derivando a w con respecto a t, se tiene

$$\frac{\partial w}{\partial t} = exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right].$$

$$\left[\frac{x}{2}\left(1+\frac{1}{t^2}\right)+y\left(t\tau+\frac{1}{t^3\tau}\right)+\frac{3z}{2}\left(t^2\delta+\frac{1}{t^4\delta}\right)\right]$$

$$= w\left[\frac{x}{2}\left(1+\frac{1}{t^2}\right)+y\left(t\tau+\frac{1}{t^3\tau}\right)+\frac{3z}{2}\left(t^2\delta+\frac{1}{t^4\delta}\right)\right]$$

sustituyendo w por su desarrollo en serie dado en (17),

$$\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} t^n J_n(x,y,z;\tau,\delta) = \frac{x}{2}\left[\sum_{n=-\infty}^{\infty} t^{n} J_n(x,y,z;\tau,\delta) + \sum_{n=-\infty}^{\infty} t^{n-2} J_n(x,y,z;\tau,\delta)\right]$$

$$+ y\left[\tau\sum_{n=-\infty}^{\infty} t^{n+1} J_n(x,y,z;\tau,\delta) + \frac{1}{\tau}\sum_{n=-\infty}^{\infty} t^{n-3} J_n(x,y,z;\tau,\delta)\right]$$

$$+ \frac{3z}{2}\left[\delta\sum_{n=-\infty}^{\infty} t^{n+2} J_n(x,y,z;\tau,\delta) + \frac{1}{\delta}\sum_{n=-\infty}^{\infty} t^{n-4} J_n(x,y,z;\tau,\delta)\right].$$

Efectuando la derivada del lado izquierdo de la ecuación, se tiene

$$\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} t^n J_n(x,y,z;\tau,\delta)=\sum_{n=-\infty}^{\infty} nt^{n-1}J_n(x,y,z;\tau,\delta)$$

$$=\sum_{n=-\infty}^{\infty} t^n(n+1)J_{n+1}(x,y,z;\tau,\delta)=A$$

donde hemos aplicado un cambio de índice.

Entonces,

$$A=\frac{x}{2}\left[\sum_{n=-\infty}^{\infty} t^n J_n(x,y,z;\tau,\delta)+\sum_{n=-\infty}^{\infty} t^n J_{n+2}(x,y,z;\tau,\delta)\right]$$

$$+y\left[\tau\sum_{n=-\infty}^{\infty} t^n J_{n-1}(x,y,z;\tau,\delta)+\frac{1}{\tau}\sum_{n=-\infty}^{\infty} t^n J_{n+3}(x,y,z;\tau,\delta)\right]$$

$$+\frac{3z}{2}\left[\delta\sum_{n=-\infty}^{\infty} t^n J_{n-2}(x,y,z;\tau,\delta)+\frac{1}{\delta}\sum_{n=-\infty}^{\infty} t^n J_{n+4}(x,y,z;\tau,\delta)\right] \quad (18)$$

Comparando, y luego haciendo el cambio $n=n-1$,se llega a la relación (16).

Otras relaciones de recurrencia

$$\frac{\partial}{\partial x}[x^n J_n(x,y,z;\tau,\delta)]=x^{n-1}\Big[nJ_n(x,y,z;\tau,\delta)+$$

$$\frac{x}{2}[J_{n-1}(x,y,z;\tau,\delta)-J_{n+1}(x,y,z;\tau,\delta)]\Big] \quad (19)$$

$$\frac{\partial}{\partial y}[y^n J_n(x,y,z;\tau,\delta)]=y^{n-1}\Big[nJ_n(x,y,z;\tau,\delta)+$$

$$\frac{y}{2}\left[\tau J_{n-2}(x,y,z;\tau,\delta)-\frac{1}{\tau}J_{n+2}(x,y,z;\tau,\delta)\right]\Big] \quad (20)$$

$$\frac{\partial}{\partial z}\left[z^n J_n(x,y,z;\tau,\delta)\right] = z^{n-1}\Big[n J_n(x,y,z;\tau,\delta) +$$

$$\frac{z}{2}\left[\delta J_{n-3}(x,y,z;\tau,\delta) - \frac{1}{\delta}\,J_{n+3}(x,y,z;\tau,\delta)\right]\Big] \tag{21}$$

Demostración de (19)

Aplicando la derivada del producto en (19), se tiene

$$\frac{\partial}{\partial x}\left[x^n J_n(x,y,z;\tau,\delta)\right] = n x^{n-1} J_n(x,y,z;\tau,\delta) + x^n\,\frac{\partial}{\partial x} J_n(x,y,z;\tau,\delta) \tag{22}$$

usando (11) se obtiene (19).

Similarmente, para demostrar (20) y (21) se aplica la derivada del producto y se usan, respectivamente, (12) y (13).

Ecuación Diferencial

A continuación se presentan una ecuación diferencial que satisface las funciones de Bessel de tres variables, dos parámetros y un índice,

La función de Bessel $J_n(x,y,z;\tau,\delta)$ satisface la ecuación diferencial

$$\left[x^2\frac{\partial}{\partial x^2} - x\,\frac{\partial}{\partial x} + x^2 - n^2 + 2\tau\left[(1-2n) + 2\tau\,\frac{\partial}{\partial \tau}\right]\frac{\partial}{\partial \tau} + 3\delta\left[(1-2n) + 3\delta\,\frac{\partial}{\partial \delta}\right]\right.$$

$$\left. + 12\tau\,\delta\,\frac{\partial^2}{\partial \tau}\,\partial\delta\right] J_n = 0 \tag{23}$$

Demostración de (23)

Para mostrar el resultado (23) de (16) y las relaciones de recurrencia (14) y (15), se tiene

$$n J_n(x,y,z;\tau,\delta) = \frac{x}{2}\left[J_{n-1}(x,y,z;\tau,\delta) + J_{n+1}(x,y,z;\tau,\delta)\right] + 2\tau\,\frac{\partial}{\partial \tau} J_n(x,y,z;\tau,\delta)$$

$$+ 3\delta\,\frac{\partial}{\partial \delta} J_n(x,y,z;\tau,\delta) \tag{24}$$

Despejando $\frac{1}{2}\,J_{n-1}(x,y,z;\tau,\delta)$ con $x \neq 0$

$$\frac{n}{x} J_n(x, y, z; \tau, \delta) - \frac{1}{2} J_{n+1}(x, y, z; \tau, \delta)$$

$$-\frac{2\tau}{x}\frac{\partial}{\partial \tau} J_n(x, y, z; \tau, \delta) - \frac{3\delta}{x}\frac{\partial}{\partial \delta} J_n(x, y, z; \tau, \delta) = \frac{1}{2} J_{n-1}(x, y, z; \tau, \delta) \tag{25}$$

De la relación de recurrencia (11), (25) puede escribirse como

$$J_{n-1}(x, y, z; \tau, \delta) = \left[\frac{n}{x} + \frac{\partial}{\partial x} - 2\frac{\tau}{x}\frac{\partial}{\partial \tau} - \frac{3\delta}{x}\frac{\partial}{\partial \delta}\right] J_n(x, y, z; \tau, \delta)$$

Definiendo el operador $S_- = \left[\frac{n}{x} + \frac{\partial}{\partial x} - 2\frac{\tau}{x}\frac{\partial}{\partial \tau} - \frac{3\delta}{x}\frac{\partial}{\partial \delta}\right]$ (26)

Despejando J_{n+1} de (25)

$$\frac{n}{x} J_n(x, y, z; \tau, \delta) - \frac{2\tau}{x}\frac{\partial}{\partial \tau} J_n(x, y, z; \tau, \delta) - 3\frac{\delta}{x}\frac{\partial}{\partial \delta} J_n(x, y, z; \tau, \delta)$$

$$-\frac{1}{2} J_{n-1}(x, y, z; \tau, \delta) = \frac{1}{2} J_{n+1}(x, y, z; \tau, \delta)$$

De la relación de recurrencia (11), se tiene

$$J_{n+1}(x, y, z; \tau, \delta) = \left[\frac{n}{x} - \frac{2\tau}{x}\frac{\partial}{\partial \tau} - 3\frac{\delta}{x}\frac{\partial}{\partial \delta} - \frac{\partial}{\partial x}\right] J_n(x, y, z; \tau, \delta) \tag{27}$$

Se define

$$S_+ = \left[\frac{n}{x} - \frac{\partial}{\partial x} - 2\frac{\tau}{x}\frac{\partial}{\partial \tau} - \frac{3\delta}{x}\frac{\partial}{\partial \delta}\right].$$

Estos operadores verifican:

$$S_+(J_n) = J_{n+1}$$

$$S_-(J_n) = J_{n-1}$$

Esto nos permite demostrar que J_n satisface

$$\left[n - 1 + x\frac{\partial}{\partial x} - 2\tau\frac{\partial}{\partial \tau} - 3\delta\frac{\partial}{\partial \delta}\right]\left[n - x\frac{\partial}{\partial x} - 2\tau\frac{\partial}{\partial \tau} - 3\delta\frac{\partial}{\partial \delta}\right] J_n = 0$$

de la cual se obtiene (23).

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Integrales y ecuación diferencial que involucran la función de Wright

Susana Salinas de Romero[2], Daniel Meza[1] y Marleny Fuenmayor[2]

Dedicado al Dr. Shyam Kalla en su 76º aniversario

[1]Universidad Popular del César.

[2]Centro de Investigación de Matemática Aplicada (C.I.M.A.). Facultad de Ingeniería. Universidad del Zulia

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Resumen

Este artículo está basado en el estudio de la función de Wright $p\Psi q$, la función hipergeométrica generalizada pFq y el polinomio general $S_q^p[x]$. Por último, como un ejemplo de aplicación a la función de Wright, se considera el problema de obtener la solución de la ecuación del calor de segundo orden con condición en la frontera de una barra uniforme.

Palabras clave: Función H de Fox, función de Wright, la función pFq, y ecuación de calor unidimensional.

Integrals and differential equations involving Wright functions

Abstract

This article is based on the study of Wright's function, hypergeometric generalized's pFq function and the general polynomial $S_q^p[x]$. Finally as an application example of Wright function is considered the problem to obtain the solution of the heat second-order differential equation condition, on a uniform rod.

Key words: H-Fox's function, Wright's function, pFq, function, one dimensional heat equation.

Introducción

En las matemáticas aplicadas tienen gran relevancia las funciones especiales, ya que estas aparecen en la solución de ecuaciones diferenciales. Entre la funciones especiales mas importante están las funciones hipergeométricas: la función de Gauss, la función hipergeométrica generalizada ${}_pF_q$, la función de Appell, la función de Humbert, la función de Lauricella, la función H de Fox, [6,7], la función de Wright, entre otras. Estas se encuentran en muchas aplicaciones en estadística, teoría cuántica, ecuaciones funcionales, vibración de vigas, conducción de calor, elasticidad, radiación, y en general, en aplicaciones a la ingeniería, etc. Debido a su importancia se estudian sus propiedades, desarrollos asintóticos, desarrollos en serie, etc., a fin de obtener un estudio detallado del comportamiento analítico de tales funciones.

El objetivo de este trabajo de investigación es evaluar algunas integrales que envuelven algunas funciones especiales entre ellas la función de Wright, la función pFq y resolver la ecuación del calor de segundo orden unidimensional con condiciones de frontera que envuelven la función de Wright. Otros autores han evaluado funciones integrales que involucran funciones especiales [2-10,12], y han presentado otras generalizaciones de las funciones hipergeométricas y sus propiedades [9,11,13-19]. Una característica importante de la función de Wright $p\Psi q$ [15, 19-21] es que generaliza la función hipergeométrica usual $_pF_q$, lo cual hace que los resultados obtenidos de $p\Psi q$ sean de interés.

La Función Hipergeométrica Generalizada de Wright

Una interesante generalización de la serie $p\Psi q$ fue introducida por el matemático E. M. Wright (1935 a 1940) quien estudió la expansión asintótica de la función generalizada definida por: [13, p. 21, No. (38)]

$$p\Psi q\left[\begin{matrix}(\alpha_1, A_1),(\alpha_2, A_2),...,(\alpha_p, A_p);\\(\beta_1, B_1),(\beta_2, B_2),...,(\beta_q, B_q);\end{matrix} z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j{}^n)z^n}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j{}^n)n!}, \quad (1)$$

donde los coeficientes, $A_1, A_2,..., A_p$ y $B_1, B_2,..., B_q$ son números reales positivos tales que: [14, p. 21, No. (39)]

$$1+\sum_{j=1}^{q}B_j - \sum_{j=1}^{p}A_j \geq 0 \text{ donde } A_p, B_q \in \Re^+$$

Si comparamos la definición [14, p. 21, No. (40)], tenemos:

$$p\Psi q\left[\begin{matrix}(\alpha_1,1),(\alpha_2,1),...,(\alpha_p,1);\\(\beta_1,1),(\beta_2,1),...,(\beta_q,1);\end{matrix} z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)z^n}{\prod_{j=1}^{q}\Gamma(\beta_j)n!}\,{}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p;\\\beta_1,...,\beta_q;\end{matrix} z\right], \quad (2)$$

El polinomio $S_q^p[x]$ se define como [8, eq., (16)]

$$S_q^p[x]=\sum_{k=0}^{[q/p]}\frac{(-q)_{p,k}}{k!}F_{q,k}x^k, q=0,1,2,... \quad (3)$$

donde p es un entero positivo arbitrario y los coeficientes de $F_{q,k}(q,k)\geq 0$ son constantes reales o complejas arbitrarias. Al especificar adecuadamente los coeficientes de $F_{q,k}$, el polinomio $S_q^p[x]$ puede ser reducido, a polinomios clásicos ortogonales conocidos, tales como Jacobi, Hermite, Legendre, Chebyshev y polinomios de Laguerre.

Los siguientes resultados son obtenidos por Chaurasia [3]

$$\int_0^L\left(sen\frac{\pi x}{l}\right)^{w-1} sen\frac{\lambda_m \pi x}{L}dx = L2^{1-w} sen\frac{\lambda_m \pi x}{2}\frac{\Gamma(w)}{\Gamma\left(\frac{w\mp\lambda_m+1}{2}\right)}; \qquad \text{Re}(w)>0 \quad (4)$$

$$\int_0^L\left(sen\frac{\pi x}{l}\right)^{w-1} \cos\frac{\lambda_m \pi x}{L}dx = L2^{1-w} \cos\frac{\lambda_m \pi x}{2}\frac{\Gamma(w)}{\Gamma\left(\frac{w\mp\lambda_m+1}{2}\right)}; \qquad \text{Re}(w)>0 \quad (5)$$

Integrales que involucran a la Función de Wright

Consideremos la siguiente integral

$$I = \int_{-1}^{1} (1+t)^{2\rho-1} (1-t)^{2\nu-1} \, p\Psi q \left[\begin{matrix} (\lambda_1,1),\ldots,(\lambda_p,1); & \frac{(1-t^2)^{2i}}{(1+t^2)^{2i}} \\ (u_1,1),\ldots,(u_q,1); & \end{matrix} \right] dt \tag{6}$$

Usando la definición de la función de Wright, dada por la ecuación (1), se tiene

$$\int_{-1}^{1} (1+t)^{2\rho-1} (1-t)^{2\nu-1} \left[\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\lambda_j + n)}{\prod_{j=1}^{q} \Gamma(u_j + n) n!} (1-t^2)^{2ni} (1+t^2)^{-2ni} \right] dt$$

Intercambiando el orden de integración y la suma en base a la convergencia absoluta [1],

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\lambda_j + n)}{\prod_{j=1}^{q} \Gamma(u_j + n) n!} \int_{-1}^{1} (1+t)^{2\rho-1} (1-t)^{2\nu-1} (1-t^2)^{-\rho-\nu} (1+t^2)^{-2ni} dt \tag{7}$$

Sea

$$I_1 = \int_{-1}^{1} (1+t)^{2\rho-1} (1-t)^{2\nu-1} (1+t^2)^{-\rho-\nu-2.} \; (1-t^2)^{2} \; dt .$$

Haciendo en esta integral el cambio de variable y realizando las operaciones respectivas, se obtiene

A

$$I_1 = \int_{-1/2}^{\infty} \left(1+\frac{u}{1+u}\right)^{2\rho-1} \left(1-\frac{u}{1+u}\right)^{2\nu-1} \left(1+\left(\frac{u}{1+u}\right)^2\right)^{-\rho-\nu-2ni} \left(1-\left(\frac{u}{1+u}\right)^2\right)^{2ni} \frac{1}{(1+u)^2} du . \tag{8}$$

Luego, se tiene

$$I_1 = \int_{-1/2}^{\infty} (1+2u)^{2\rho-1} (1+2u+2u^2)^{-\rho-\nu-2ni} (1+2u)^{2ni} du .$$

Teniendo en cuenta un nuevo cambio de variable $1+2u = \tan\theta$ y las operaciones respectivas, con el cambio de límites de integración, se obtiene

$$I_1 = 2^{\rho+\nu+2ni-1} \int_{0}^{\pi/2} sen^{\rho+\nu+2ni-1}(\theta) \cos^{\rho+\nu+2ni-1}(\theta) d\theta$$

Luego, usando la definición de la función beta, se tiene

$$I_1 = 2^{\rho+\nu+2ni-1} \frac{\Gamma(\rho+ni)\Gamma(\nu+ni)}{\Gamma(\rho+2ni+\nu)} \tag{9}$$

Sustituyendo la ecuación (9) en (7), se obtiene

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\lambda_j+n)}{\prod_{j=1}^{q}\Gamma(u_j+n)n!}2^{\rho+v+2ni-1}\frac{\Gamma(\rho+ni)\Gamma(v+ni)}{\Gamma(\rho+2ni+v)}$$

$$=2^{\rho+v-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\lambda_j+n)}{\prod_{j=1}^{q}\Gamma(u_j+n)n!}\frac{\Gamma(\rho+ni)\Gamma(v+ni)}{\Gamma(\rho+v+2ni)}4^{ni}$$

Usando la representación en serie para $p\Psi q$, se tiene

$$=2^{\rho+v+2ni-2}\,{}_{(p+2i)}\Psi_{q+i}\left[\begin{matrix}(\lambda_1,1),\ldots,(\lambda_p,1),(\rho,i)(v,i);\\(u_1,1),\ldots,(u_q,1),(\rho+v,2i);\end{matrix}4^i\right]$$

donde $i\in N$, $\mathrm{Re}(\rho)$, $\mathrm{Re}(v)>0$ $\quad u_i\neq 0,-1,-2,\ldots,$ $\quad i=1,2,\ldots,q$

Evaluación de Integrales con la Función de Wright

Considerando la siguiente integral

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(ax^2+bxy+cy^2)}p\Psi q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}c\right]dxdy$$

$$=\frac{2\pi}{\sqrt{4ac-b^2}}p\Psi q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}c\right].$$

Usando la definición de la función de Wright, dada por la ecuación (1), se tiene

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(ax^2+bxy+cy^2)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a+A_jn)}{\prod_{j=1}^{q}\Gamma(b+B_jn)n!}c^n dxdy. \quad (10)$$

En virtud de que la serie converge absolutamente, es posible intercambiar la suma con la integral [1]

$$I=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a+A_jn)}{\prod_{j=1}^{q}\Gamma(b+B_jn)n!}c^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(ax^2+bxy+cy^2)}dxdy. \quad (11)$$

Sea $I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(ax^2+bxy+cy^2\right)}dxdy.$

Efectuando, la siguiente sustitución, utilizando la definición $\Gamma(1/2) = \sqrt{\pi}$, se obtiene

$$u = \sqrt{a}x + \frac{by}{2\sqrt{a}}, \quad y \quad \frac{du}{\sqrt{a}} = dx \;;\; I_1 = \frac{2\pi}{\sqrt{4ac-b^2}}, \tag{12}$$

Reemplazando y usando la representación en serie de (1), se tiene

$$I = \frac{2\pi}{\sqrt{4ac-b^2}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(a+A_j n\right)}{\prod_{j=1}^{q}\Gamma\left(b+B_j n\right)n!}c^n,$$

$$= \frac{2\pi}{\sqrt{4ac-b^2}}\, p\Psi q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\; c\right]. \tag{13}$$

$\mathrm{Re}(a), \mathrm{Re}(b) > 0, \qquad \mathrm{Re}(4ac - b^2) > 0.$

Caso Particular

En este trabajo se obtienen, como caso particular de la integral generalizada, la siguiente integral con funciones hipergeométricas ${}_pF_q(z)$:

haciendo $A_1 = A_2 =,\ldots,= A_p = 1; \qquad B_1 = B_2 =,\ldots,= B_q = 1$ y usando la definición (2),

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(ax^2+bxy+cy^2\right)}\,{}_pF_q\left[\begin{matrix}(a_1),\ldots,(a_p);\\(b_1),\ldots,(b_q);\end{matrix}\; c\right]dxdy$$

$$= \frac{2\pi}{\sqrt{4ac-b^2}}\,{}_pF_q\left[\begin{matrix}(a_1),\ldots,(a_p);\\(b_1),\ldots,(b_q);\end{matrix}\; c\right].$$

$\mathrm{Re}(a), \mathrm{Re}(b) > 0, \qquad \mathrm{Re}(4ac - b^2) > 0.$

Un problema de valores iniciales con condiciones que involucran la función de Wright

Como ejemplo de la aplicación de la función Wright multivariable de la matemática aplicada, vamos a considerar el problema de flujo de calor en una barra uniforme con condiciones de contorno, es decir, el problema de la conducción de calor en la barra uniforme con la condición Robin a temperatura cero, con la radiación en los extremos dentro de la media. Usando la función de Wright (1) y una clase de polinomio general (3). Si el coeficiente térmico es constante y no hay ninguna fuente de energía térmica, la $u(x,t)$ que representa la temperatura en una varilla de longitud L, $0 < x < L$, y u debe satisfacer la ecuación del calor.

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < L \tag{14}$$

Sujeto a la condición inicial:

$$u(x,0)=\left(sen\frac{\pi x}{L}\right)^{w-1} S_q^p\left[\left(sen\frac{\pi x}{L}\right)^{2\sigma}\right] p\Psi q\left[\begin{matrix}(\alpha_1,A_1),\dots,(\alpha_p,A_p);\\(\beta_1,B_1),\dots,(\beta_q,B_q);\end{matrix}\left(sen\frac{\pi x}{L}\right)^{2}\right] \quad (15)$$

Y a la condición de frontera

$$\frac{\partial u}{\partial t}(0,t)-hu(0,t)=0, \quad t>0, \quad h>0 \tag{16}$$

$$\frac{\partial u}{\partial t}(L,t)-hu(L,t)=0, \quad t>0, \quad h>0 \tag{17}$$

Solucionando la ecuación (14) por el método de separación de variables

$$u(x,t)=X(x)T(t)$$

Obteniendo las derivadas parciales y sustituyendo obtenemos dos ecuaciones diferenciales ordinarias lineales con coeficientes constantes y otra de variables separables cuya solución es la siguiente

$$u(x,t)=[A\cos(\lambda x)+Bsen(\lambda x)]e^{-\mu\lambda^2 t} \tag{18}$$

Aplicando la condición de frontera y realizando las operaciones respectivas se obtiene

$$B=\frac{hA}{\lambda}, \lambda\neq 0 \tag{19}$$

$$\tan(\lambda L)=\frac{2h\lambda}{(\lambda^2-h^2)}, h\neq\lambda \tag{20}$$

En donde λ_n satisfacen (20) y son infinitos valores propios. Utilizando algunos métodos numéricos como el de Newton, el de Punto fijo y el de Bisección se verificó cuál de estos da una mejor aproximación para calcular los valores propios.

Observando el método de Newton los valores generados son muy oscilantes y no dan la mejor convergencia para los valores propios de la función (20). El método Punto fijo únicamente muestra el punto inicial y las demás raíces no. Estos dos métodos no fueron consistentes.

El método que mostró mejores resultados para calcular los valores propios de (20) fue el de Bisección.

A continuación se presenta un ejemplo donde se calcula el valor propio entre el intervalo [0, 0.2] ver Fig. 1.

Resultados del método de bisección según la rutina de Matlab Versión (7.01)

Tabla 1.

iter	Valor propio	Error
1	0.10000000000000	
2	0.15000000000000	0.05000000000000
3	0.12500000000000	0.02500000000000
4	0.11250000000000	0.01250000000000
5	0.11875000000000	0.00625000000000
6	0.12187500000000	0.00312500000000
7	0.12343750000000	0.00156250000000
8	0.12265625000000	0.00078125000000
9	0.12304687500000	0.00039062500000
10	0.12324218750000	0.00019531250000
11	0.12314453125000	0.00009765625000
12	0.12319335937500	0.00004882812500
13	0.12321777343750	0.00002441406250
14	0.12320556640625	0.00001220703125
15	0.12319946289063	0.00000610351563
16	0.12320251464844	0.00000305175781
17	0.12320404052734	0.00000152587891
18	0.12320327758789	0.00000076293945
19	0.12320289611816	0.00000038146973
20	0.12320270538330	0.00000019073486

La raíz que se obtuvo con una tolerancia de 10^{-7} es 0.12320270538330

Figura No 1

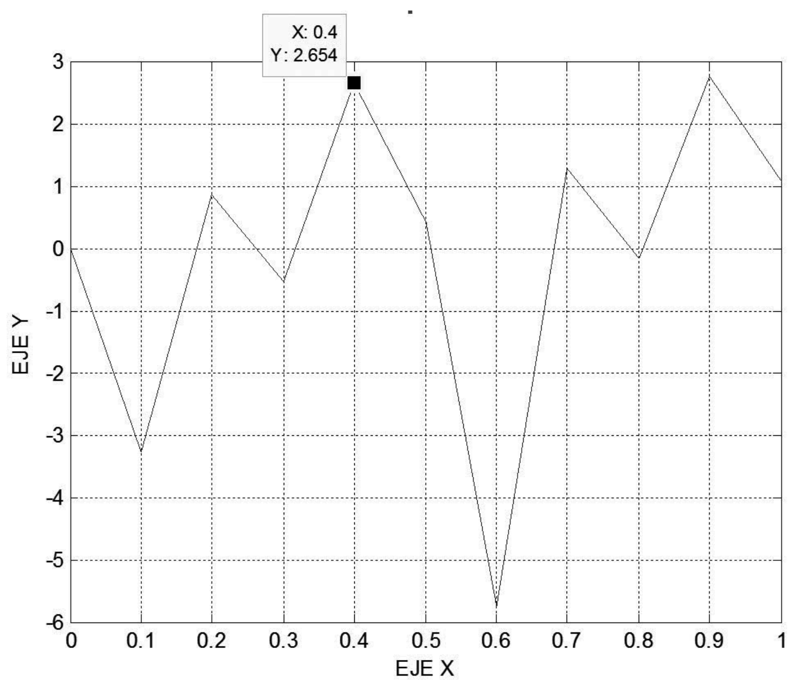

Fig. 1. Gráfica del Método de Bisección

Utilizando el principio de superposición

$$u(x,t)=\sum_{n=1}^{\infty} A_n\left[\cos(\lambda_n x)+\frac{h}{\lambda_n} sen(\lambda x)\right]e^{-\mu\lambda_n^2 t} \tag{21}$$

Aplicando las condiciones iniciales a la ecuación (20) y utilizando los conceptos de ortogonalidad, ortonormalidad para hallar A_n y las identidades trigonométricas, se obtiene el siguiente resultado:

$$A_n=\frac{2\lambda_n^2}{\left[\left(\lambda_n^2+h^2\right)L+2h\right]}\int_0^L\left[\lambda_n\cos(\lambda_n x)+hsen(\lambda_n x)\right]$$

$$\times\left(sen\frac{\pi x}{L}\right)^{w-1} S_q^p\left[\left(sen\frac{\pi x}{L}\right)^{2\sigma}\right] p\Psi q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\left(sen\frac{\pi x}{L}\right)^2\right]dx \tag{22}$$

Luego, usando (4) y (5), se tiene

$$\int_0^L\left[\lambda_n\cos(\lambda_n x)+hsen(\lambda_n x)\right]\left[\lambda_m\cos(\lambda_m x)+hsen(\lambda_m x)\right]dx$$

$$=\begin{cases}\dfrac{2\lambda_n^2}{\left[\left(\lambda_n^2+h^2\right)L+2h\right]}, & m=n\\ 0 & m\neq n\end{cases} \tag{23}$$

$$A_{n=}\int_0^L\left[\lambda_n\cos(\lambda_n x)+hsen(\lambda_n x)\right]\left(sen\frac{\pi x}{L}\right)^{w-1} S_q^p\left[\left(sen\frac{\pi x}{L}\right)^{2\sigma}\right]$$

$$.p\Psi q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\left(sen\frac{\pi x}{L}\right)^2\right]dx \tag{24}$$

Aplicando las definiciones (1) y (2) en (24) se obtiene

$$\int_0^L\left[\lambda_n\cos(\lambda_n x)+hsen(\lambda_n x)\right]\left(sen\frac{\pi x}{L}\right)^{w-1}$$

$$\sum_{k=0}^{[q/p]}\frac{(-q)_{pk}}{k!}F_{q,k}\left(sen\frac{\pi x}{L}\right)^{2\sigma k}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j n)}{\prod_{j=1}^{q}\Gamma(b_j+B_j n)}\frac{\left(sen\frac{\pi x}{L}\right)^{2k}}{n!}dx \tag{25}$$

Intercambiando el orden de la integral y la suma en base a la convergencia absoluta, se obtiene [1]

$$A_n = \sum_{k=0}^{[q/p]} \frac{(-q)_{pk}}{k!} F_{q,k} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n)} \frac{z^k}{n!}$$

$$\int_0^L [\lambda_n \cos(\lambda_n x) + h sen(\lambda_n x)] \left(sen \frac{\pi x}{L} \right)^{w-1} \left(sen \frac{\pi x}{L} \right)^{2\sigma k} \left(sen \frac{\pi x}{L} \right)^{2k} dx . \qquad (26)$$

Realizado las operaciones respectivas, teniendo en cuenta los resultados (5) y (6), y la representación en serie (1) en la ecuación (26)

$$u(x,t) = \frac{2L}{2^{w+2k(\sigma+1)-1}} \sum_{k=0}^{[q/p]} \frac{(-q)_{pk}}{k!} F_{q,k} \frac{2\lambda_n^2}{[(\lambda_n^2 + h^2)L + 2h]}$$

$$\times \left[\cos(\lambda_n x) + \frac{h}{\lambda_n} sen(\lambda_n x) \right] e^{-\mu \lambda_n^2 t} \left[\lambda_n \cos\left(\frac{\lambda_n \pi}{2} \right) + h sen\left(\frac{\lambda_n \pi}{2} \right) \right]$$

$$\times \frac{\Gamma(w + 2k(\sigma+1))}{\Gamma(w \pm 2k(\sigma+1) + \lambda_n + 1)} p\Psi q \left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{matrix} 1 \right] \qquad (27)$$

$$\sigma > 0, k > 0$$

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Desigualdades integrales fraccionales y sus *q*-análogos

Sunil Dutt Purohit[1], Faruk Uçar[2] and R.K. Yadavc[3]

Dedicated to Professor S.L. Kalla

[1] Department of Basic Sciences (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology, Udaipur-313001, Rajasthan, India.
sunil_a_purohit@yahoo.com

[2] Department of Mathematics, University of Marmara, TR-34722, Kad köy, Istanbul, Turkey.
fucar@marmara.edu.tr

[3] Department of Mathematics and Statistics, J. N. Vyas University. Jodhpur-342005, India.
rkmdyadav@gmail.com

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Resumen

El objeto de este trabajo es establecer algunas desigualdades que envuelven operadores integrales de Saigo. Se usa el calculo q-fraccional para obtener varios resultados en la teoría de las desigualdades q-integrales. Los resultados dados anteriormente por Purohit y Raina (2013) y Sulaiman (2011) son casos especiales de los obtenidos en este trabajo.

Palabras clave: Desigualdades integrales, operadores integrales fraccionales, operadores q-integrales fraccionales.

On fractional integral inequalities and their *q*-analogues

Abstract

The aim of this paper is to establish some integral inequalities involving Saigo fractional integral operators. We then use fractional q-calculus for yielding various results in the theory of q-integral inequalities. The results given earlier by Purohit and Raina (2013) and Sulaiman (2011) follow as special cases of our findings.

Key words: Integral inequalities, fractional integral operators, fractional q-integral operators.

Introduction

Fractional integral inequalities have many applications, the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems, and in fractional partial differential equations. Further, they also provide upper and lower bounds to the solutions of the above equations. For detailed applications, one may refer to the book [1], and the recent papers [2]-[5] on the subject.

In a recent paper, Purohit and Raina [6] investigated certain Chebyshev type ([7]) integral inequalities involving the Saigo fractional integral operators, and also established the *q*-extensions of the main results. The aim of this paper is to establish several new integral inequalities for synchronous functions that are related to the Chebyshev functional using the Saigo fractional integral. *q*-Extensions of the main results are also established. Some of the results due to Purohit and Raina [6] and Sulaiman [8] follows as special cases of our results.

Following definitions will be needed in the sequel.

Definition 1. Two functions f and g are said to be synchronous on $[a,b]$, if

$$\{(f(x)-f(y))(g(x)-g(y))\} \geq 0, \tag{1}$$

for any $x, y \in [a,b]$.

Definition 2. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space $(C_\mu$ $(\mu \in R)$, if there exists a real number $p > \mu$ such that $f(t) = t^p\,\phi(t)$; where $\phi(t) \in C\,(0, \infty)$.

Definition 3. Let $\alpha > 0$, $\beta,\eta \in R$, then the Saigo fractional integral $I^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by ([9] see also [10, p. 19], [11]):

$$I_{0\,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right) f(\tau)d\tau, \tag{2}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (2) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{t^n}{n!}, \tag{3}$$

and (t) $(a)_n$ is the Pochhammer symbol

$$(a)n = a(a+1)\cdots(a+n-1),\ (a)_0 = 1.$$

The integral operator (2) includes both the Riemann-Liouville and the Erde'lyi-Kober fractional integral operators given by the following relationships:

$$R^\alpha\{f(t)\} = I_{0\,t}^{\alpha,-\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \quad (\alpha > 0) \tag{4}$$

and

$$I^{\alpha,\eta}\{f(t)\} = I_{0\,t}^{\alpha,0,\eta}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^\eta f(\tau)d\tau \quad (\alpha > 0, \eta \in R). \tag{5}$$

For $f(t)=t^{\mu}$ in (2), we get the known result [9]:

$$I_{0,t}^{\alpha,\beta,\eta}\left\{t^{\mu}\right\}=\frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)}\,t^{\mu-\beta}, \tag{6}$$

$$(\alpha>0, min(\mu,\mu-\beta+\eta)>-1, t>0)$$

which shall be used in the sequel.

Fractional Integral Inequalities

The following theorems involving Saigo integral inequalities for the synchronous functions will be established.

Theorem 1. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$, $h>0$, *then for all* $t>0$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$).

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}}\,I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)g(t)h(t)\right\}\geq I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{g(t)h(t)\right\}+$$

$$I_{0,t}^{\alpha,\beta,\eta}\left\{g(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)\,h(t)\right\}-I_{0,t}^{\alpha,\beta,\eta}\left\{h(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)\,g(t)\right\}. \tag{7}$$

Proof: Using Definition 1 and $h>0$, for all $\tau,\rho\geq 0$, we have

$$\left\{(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)+h(\rho))\right\}\geq 0 \tag{8}$$

which implies that

$$f(\tau)g(\tau)h(\tau)+f(\rho)g(\rho)h(\rho)\geq f(\tau)g(\rho)h(\rho)+f(\rho)g(\tau)h(\tau)+g(\tau)f(\rho)h(\rho)$$

$$+g(\rho)f(\tau)h(\tau)-h(\tau)f(\rho)g(\rho)-h(\rho)f(\tau)g(\tau) \tag{9}$$

Consider

$$F(t,\tau)=\frac{t^{-\alpha-\beta}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\quad(\tau\in(0,t);t>0) \tag{10}$$

$$=\frac{1}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}}+\frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}}+$$

$$\frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}}+\cdots.$$

Since each term of the above series is positive in view of the conditions stated with Theorem 1, we observe that the function $F(t,\tau)$ remains positive, for all $\tau \in (0,t)$ $(t > 0)$.

Multiplying both sides of (9) by $F(t,\tau)$ (defined above by (10)) and integrating with respect to τ from 0 to t, and using (2), we get

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} + f(\rho)g(\rho)h(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{1\} \geq g(\rho)h(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} +$$

$$f(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\} + f(\rho)h(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{g(t)\} + g(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\} -$$

$$f(\rho)g(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{h(t)\} - h(\rho)\, I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}. \qquad (11)$$

Next, multiplying both sides of (11) by $F(t,\rho)$ $(\rho \in (0,t), (t > 0)$, where $F(t,\rho)$ is given by (10), and integrating with respect to ρ from 0 to t, and using formula (6), we arrive at the desired result (7).

Theorem 2. *Let f and g be two synchronous functions on $[0,\infty)$, and $h > 0$, then*

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}}\, I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} + \frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\, t^{\delta}}\, I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\} + I_{0,t}^{\alpha,\beta,\eta}\{g(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\} - I_{0,t}^{\alpha,\beta,\eta}\{h(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\},$$
(12)

for all $t > 0, \alpha > \max\{0,-\beta\}, \gamma > \max\{0,-\delta\}, \beta, \delta < 1, \beta - 1 < \eta < 0, \delta - 1 < \zeta < 0.$

Proof: To prove the above theorem, we start with the inequality (11). On multiplying both sides of (11) by

$$\frac{t^{-\gamma-\delta}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)}\, {}_2F_1\left(\gamma+\delta, -\zeta; \gamma; 1-\frac{\rho}{t}\right) \quad (\rho \in (0,t); t > 0),$$

and taking integration with respect to ρ from 0 to t, we get

$$I_{0,t}^{\gamma,\delta,\zeta}\{1\} I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} + I_{0,t}^{\alpha,\beta,\eta}\{1\}\, I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} \geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\} + I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\} - I_{0,t}^{\alpha,\beta,\eta}\{h(t)\} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}\, I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\},$$

which on using (6) readily yields the desired result (12).

Remark 1. It may be noted that the inequalities (7) and (12) are reversed if the functions are asynchronous on $[0,\infty)$ i.e.

$$\{(f(x)-f(y))(g(x)-g(y))\}\leq 0, \tag{13}$$

for any $x, y\in[0,\infty)$.

Remark 2. For $\gamma=\alpha,\ \delta=\beta, \zeta=\eta$, Theorem 2 immediately reduces to Theorem 1.

Theorem 3. *Let* f *,* g *and* h *be three monotonic functions on* $[0,\infty)$ *satisfying the inequality*

$$\{(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))\}\geq 0, \tag{14}$$

then for all , $t>0, \alpha>\max\{0,-\beta\}, \gamma>\max\{0,-\delta\}, \beta, \delta<1, \beta-1<\eta<0, \delta-1<\zeta<0.$

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} \ \frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\ t^{\delta}} I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}I_{0,t}^{\gamma,\delta\,\mathrm{a},\zeta}\{g(t)h(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\}\ I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}-$$

$$I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)h(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{h(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\}.$$
(15)

Proof: By applying the similar procedure as of Theorem 1 and 2, one can easily establish the above theorem. Therefore, we omit the details of the proof of this theorem.

Observe that, if we set $\beta=0$ (and $\delta=0$ additionally for Theorem 2), and make use of the relation (5), Theorems 1 to 3 respectively yield the following integral inequalities involving the Erde'lyi-Kober type fractional integral operator defined by (5):

Corollary 1. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$*, and* $h>0$*, then*

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta}\ \{f(t)g(t)h(t)\}\geq I^{\alpha,\eta}\ \{f(t)\}I^{\alpha,\eta}\{g(t)h(t)\}+I^{\alpha,\eta}\{g(t)\}I^{\alpha,\eta}\{f(t)h(t)\}$$

$$-I^{\alpha,\eta}\{h(t)\}I^{\alpha,\eta}\{f(t)g(t)\}, \tag{16}$$

for all $t>0, \alpha>0, -1<\eta<0.$

Corollary 2. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$*, and* $h>0$*, then for all* $t>0, \alpha, \gamma>0, -1<\max(\eta,\zeta)<0,$

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\gamma,\zeta}\{f(t)g(t)h(t)\}+\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}\ I^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I^{\alpha,\eta}\{f(t)\}I^{\gamma,\zeta}\{g(t)h(t)\}+I^{\alpha,\eta}\{g(t)h(t)\}I^{\gamma,\zeta}\{f(t)\}+I^{\alpha,\eta}\{g(t)\}I^{\gamma,\zeta}\{f(t)h(t)\}+$$

$$I^{\alpha,\eta}\{f(t)h(t)\}I^{\gamma,\zeta}\{g(t)\}-I^{\alpha,\eta}\{h(t)\}I^{\gamma,\zeta}\{f(t)g(t)\}-I^{\alpha,\eta}\{f(t)g(t)\}I^{\gamma,\zeta}\{h(t)\}. \quad (17)$$

Corollary 3. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality* (14) *then for all* $t>0,\alpha,\gamma>0,-1<\max(\eta,\zeta)<0$,

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}I^{\gamma,\zeta}\{f(t)g(t)h(t)\}-\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}I^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I^{\alpha,\eta}\{g(t)h(t)\}I^{\gamma,\zeta}\{f(t)\}-I^{\alpha,\eta}\{f(t)\}I^{\gamma,\zeta}\{g(t)h(t)\}+I^{\alpha,\eta}\{f(t)h(t)\}I^{\gamma,\zeta}\{g(t)\}$$

$$-I^{\alpha,\eta}\{g(t)\}I^{\gamma,\zeta}\{f(t)h(t)\}-I^{\alpha,\eta}\{h(t)\}I^{\gamma,\zeta}\{f(t)g(t)\}+I^{\alpha,\eta}\{f(t)g(t)\}I^{\gamma,\zeta}\{h(t)\}. \quad (18)$$

Again, if we replace β by $-\alpha$ and δ by $-\gamma$ in Theorems 2 and 3, and make use of the relation (4), we obtain known results due to Sulaiman [8, pp. 24-25, Theorems 2.1 to 2.2].

q-Extensions of Main Results

In this section, we establish q-extensions of the results derived in the previous section. We begin with the mathematical preliminaries of q-series and q-calculus. For more details of q-calculus and fractional q-calculus one can refer to [12] and [13].

The q-shifted factorial is defined for $\alpha,q\in\mathbb{C}$ as a product of n factors by

$$(\alpha;q)_n=\begin{cases}1 & ; \quad n=0\\(1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & ; \quad n\in\mathbb{N},\end{cases} \quad (19)$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n=\frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n>0), \quad (20)$$

where the q-gamma function is defined by ([12, p. 16, eqn. (1.10.1)])

$$\Gamma_q(t)=\frac{(q;q)_\infty(1-q)^{1-t}}{(q^t;q)_\infty} \quad (0<q<1). \quad (21)$$

We note that

$$\Gamma_q(1+t)=\frac{(1-q^t)\Gamma_q(t)}{1-q}, \quad (22)$$

and if $|q|<1$, the definition (19) remains meaningful for $n=\infty$, as a convergent infinite product given by

$$(\alpha;q)_\infty=\prod_{j=0}^{\infty}(1-\alpha q^j). \quad (23)$$

Also, the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu (y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{24}$$

Let $t_0 \in \mathbb{R}$, then we define a specific time scale (see [14] and [15])

$$\mathbb{T}_{t_0} = \{t; t = t_0 q^n, n \text{ a non-negative integer}\} \cup \{0\}, \quad 0 < q < 1 \tag{25}$$

and for sake of convenience, we denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$ throughout this paper.

The q-derivative and q-integral of a function f defined on $\mathbb{T}$ are, respectively, given by (see [12, pp. 19, 22])

$$D_{q,t} f(t) = \frac{f(t)-f(tq)}{t(1-q)} \qquad (t \neq 0, q \neq 1) \tag{26}$$

and

$$\int_0^t f(\tau) d_q\tau = t(1-q)\sum_{k=0}^{\infty} q^k f(tq^k). \tag{27}$$

Definition 4. The Riemann-Liouville fractional q-integral operator of a function $f(t)$ of order α (due to [15], see also [13]) is given by

$$I_q^\alpha\{f(t)\} = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} f(\tau)\, d_q\tau \qquad (\alpha > 0, 0 < q < 1), \tag{28}$$

where

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha;q)_\infty} \quad (\alpha \in \mathbb{R}) \tag{29}$$

Definition 5. For $\alpha > 0$, $\eta \in \mathbb{R}$ and $0 < q < 1$, the basic analogue of the Kober fractional integral operator (cf. [16], [13]) is given by

$$I_q^{\alpha,\eta}\{f(t)\} = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} \tau^\eta f(\tau)\, d_q\tau. \tag{30}$$

Definition 6. For $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$, a basic analogue of the Saigo's fractional integral operator ([17, p. 172, eqn. (2.1)]) is given for $|\tau/t| < 1$ by

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1}$$

$$\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} q^{(\eta-\beta)m} (-1)^m \, q^{-m(m-1)/2} \left(\frac{\tau}{t}-1\right)_m f(\tau)\, d_q\tau, \tag{31}$$

which in view of (27), can be written as (see [17, p. 173, eqn. (2.5)]):

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = t^{-\beta}(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} q^{(\eta-\beta+1)\,m}$$

$$\times \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+m};q)_k}{(q;q)_k} \; f(tq^{k+m}). \tag{32}$$

In the sequel, we shall be using the following image formula ([17, p. 173, eqn. (2.11)]):

$$I_q^{\alpha,\beta,\eta}\{t^{\mu}\} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu+1-\beta+\eta)}{\Gamma_q(\mu+1-\beta)\Gamma_q(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{33}$$

$$(\alpha > 0, 0 < q < 1, \min(\mu, \mu - \beta + \eta) > -1, t > 0).$$

Now, we shall establish new q-integral inequalities for the synchronous functions involving the fractional q-integral operators, which can be treated as the q-analogues of the inequalities (7), (12) and (15).

Theorem 4. *Let f and g be two synchronous functions, and $h(t) > 0$ on T, then*

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\, t^{\beta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\} I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}, \tag{34}$$

where $t > 0, 0 < q < 1, \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0)$.

Proof: By the hypothesis, the functions f and g are synchronous functions on T for all $\tau, \rho \geq 0$, and $h(t) > 0$ therefore the inequality (9) is satisfied, that is

$$f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) \geq f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + g(\tau)f(\rho)h(\rho)$$

$$+ g(\rho)f(\tau)h(\tau) - h(\tau)f(\rho)g(\rho) - h(\rho)f(\tau)g(\tau)$$

Since, $\tau \in (0,t)$ $(t > 0)$, $0 < q < 1$, then $tq^{k+m} \in (0, t)$ for $k, m \in N$, therefore, on replacing τ by tq^{k+m} *in the above inequality we get*

$$f(tq^{k+m})g(tq^{k+m})h(tq^{k+m}) + f(\rho)\, g(\rho)\, h(\rho) \geq f(tq^{k+m})\, g(\rho)\, h(\rho) +$$

$$f(\rho)g(tq^{k+m})h(tq^{k+m})+g(tq^{k+m})f(\rho)h(\rho)+g(\rho)f(tq^{k+m})h(tq^{k+m})-$$

$$h(tq^{k+m})f(\rho)g(\rho)-h(\rho)f(tq^{k+m})g(tq^{k+m}). \tag{35}$$

Consider

$$\mathsf{H}(t,q)=t^{-\beta}(1-q)^{\alpha}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{(q^{\alpha+m};q)_k}{(q;q)_k}q^{(\eta-\beta+1)m+k}\,(k,m\in N). \tag{36}$$

Evidently, under the conditions stated with Theorem 4, we observe that the function $\mathsf{H}(t,q)$ is positive for all values of $k,m\in N$. Therefore, on multiplying both sides of (35) by $\mathsf{H}(t,q)$ and taking summations between the limits $k = 0$ *to* ∞, we get

$$\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})g(tq^{k+m})h(tq^{k+m})+f(\rho)g(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)\geq$$

$$g(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})+f(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)g(tq^{k+m})h(tq^{k+m})+$$

$$f(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)g(tq^{k+m})+g(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})h(tq^{k+m})-$$

$$f(\rho)g(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)h(tq^{k+m})-h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})g(tq^{k+m}).$$

Now, on again taking summation from $m = 0$ *to* ∞, and then making use of the definition (32), we obtain

$$I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\}+f(\rho)g(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{1\}\geq g(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)\}+$$

$$f(\rho)\,I_q^{\alpha,\beta,\eta}\{g(t)h(t)\}+f(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{g(t)\}+g(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)h(t)\}-$$

$$f(\rho)g(\rho)\,I_q^{\alpha,\beta,\eta}\{h(t)\}-h(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}. \tag{37}$$

Next, in the above inequality on replacing ρ by tq^{k+m}, multiplying both sides of by $\mathsf{H}(t,q)$, taking summations between the limits $k = 0$ *to* ∞, and $m = 0$ *to* ∞, and then making use of the definitions (32) and (33), we arrive at the desired inequality (34).

Theorem 5. *Let* f *and* g *be two synchronous functions on* T*, and* $h>0$*, then for all* $t > 0, 0 < q < 1$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta, \delta < 1$, $\beta-1 < \eta < 0$, $\delta-1 < \xi < 0$,

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} + \frac{\Gamma_q(1-\delta+\zeta)}{\Gamma_q(1-\delta)\Gamma_q(1+\gamma+\zeta)\,t^{\delta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{f(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}I_q^{\gamma,\delta,\zeta}\{h(t)\}. \tag{38}$$

Proof: To prove the above theorem, we start with the inequality (37). On replacing ρ by tq^{k+m} and multiplying both sides by a positive function $\mathsf{F}(t,q)$, given by

$$\mathsf{F}(t,q) = t^{-\delta}(1-q)^{\gamma}\,\frac{(q^{\gamma+\delta};q)_m(q^{-\zeta};q)_m}{(q^{\gamma};q)_m(q;q)_m}\,\frac{(q^{\gamma+m};q)_k}{(q;q)_k}\,q^{(\zeta-\delta+1)m+k}\,(k,m\in N). \tag{39}$$

taking summations between the limits $k = 0$ *to* ∞ and $m = 0$ *to* ∞, and then making use of the definition (32), then the inequality (37) leads to

$$I_q^{\gamma,\delta,\zeta}\{1\}I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{1\}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} \geq$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{f(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}I_q^{\gamma,\delta,\zeta}\{h(t)\}, \tag{40}$$

which yields the desired result by taking (33) into account.

Remark 3. The inequalities (34) and (38) are reversed if the functions are asynchronous on T.

Remark 4. Again, when $\gamma = \alpha$, $\delta = \beta$, $\zeta = \eta$, then Theorem 5 leads to Theorem 4.

Theorem 6. *Let* f *,* g *and* h *be three monotonic functions on* $[0,\infty)$ *satisfying the inequality* (14), *then for all* $t > 0, 0 < q < 1$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta,\delta < 1$, $\beta-1 < \eta < 0, \delta-1 < \zeta < 0$, *we have*

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} -$$

$$\frac{\Gamma_q(1-\delta+\zeta)}{\Gamma_q(1-\delta)\Gamma_q(1+\gamma+\zeta)\,t^{\beta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq I_q^{\alpha,\beta,\eta}\{g(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\} -$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{f(t)h(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} -$$

$$I_q^{\alpha,\beta,\eta}\{h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}+I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}I_q^{\gamma,\delta,\zeta}\{h(t)\}. \tag{41}$$

Proof: By applying the same procedure as of Theorem 4 and 5, one can establish the above theorem. Therefore, we omit the details of the proof.

Now, if we set $\beta = 0$ (and additionally $\delta = 0$ for Theorem 5), and make use of the known result [18, p.173, eqn. (2.9)], namely

$$I_q^{\alpha,0,\eta}\{f(t)\}=I_q^{\alpha,\eta}\{f(t)\}, \tag{42}$$

Theorems 4 to 6 respectively reduce to the following q-integral inequalities involving the Erde'lyi-Kober type fractional q-integral operators:

Corollary 4. *Let f and g be two synchronous functions on T, and $h>0$, then*

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq I_q^{\alpha,\eta}\{f(t)\}I_q^{\alpha,\eta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{g(t)\}I_q^{\alpha,\eta}\{f(t)h(t)\}$$

$$-I_q^{\alpha,\eta}\{h(t)\}I_q^{\alpha,\eta}\{f(t)g(t)\}, \tag{43}$$

for all $t > 0, 0 < q < 1$, $\alpha > 0$ and $-1 < \eta < 0$).

Corollary 5. *Let f and g be two synchronous functions on T, and $h>0$, then for $t > 0$, $0 < q < 1$, $\alpha, \gamma > 0$, such that $-1 < \max(\eta, \zeta) < 0$),*

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\gamma,\zeta}\{f(t)g(t)h(t)\}+\frac{\Gamma_q(1+\zeta)}{\Gamma_q(1+\gamma+\zeta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_q^{\alpha,\eta}\{f(t)\}I_q^{\gamma,\zeta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{g(t)h(t)\}I_q^{\gamma,\zeta}\{f(t)\}+I_q^{\alpha,\eta}\{g(t)\}I_q^{\gamma,\zeta}\{f(t)h(t)\}+$$

$$I_q^{\alpha,\eta}\{f(t)h(t)\}I_q^{\gamma,\zeta}\{g(t)\}-I_q^{\alpha,\eta}\{h(t)\}I_q^{\gamma,\zeta}\{f(t)g(t)\}-I_q^{\alpha,\eta}\{f(t)g(t)\}I_q^{\gamma,\zeta}\{h(t)\}. \tag{44}$$

Corollary 6. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality* (14), *then for all $t > 0$, $\alpha, \gamma, > 0, -1 < \max(\eta, \zeta) < 0, 0 < q < 1$*

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\gamma,\zeta}\{f(t)g(t)h(t)\}-\frac{\Gamma_q(1+\zeta)}{\Gamma_q(1+\gamma+\zeta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_q^{\alpha,\eta}\{g(t)h(t)\}I_q^{\gamma,\zeta}\{f(t)\}-I_q^{\alpha,\eta}\{f(t)\}I_q^{\gamma,\zeta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{f(t)h(t)\}I_q^{\gamma,\zeta}\{g(t)\}$$

$$-I_q^{\alpha,\eta}\{g(t)\}I_q^{\gamma,\zeta}\{f(t)h(t)\}-I_q^{\alpha,\eta}\{h(t)\}I_q^{\gamma,\zeta}\{f(t)g(t)\}+I_q^{\alpha,\eta}\{f(t)g(t)\}I_q^{\gamma,\zeta}\{h(t)\}. \tag{45}$$

Further, we observe that, if we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation [17, p.173, eqn. (2.7)], namely

$$I_q^{\alpha,-\alpha,\eta}\{f(t)\} = I_q^{\alpha}\{f(t)\} \tag{46}$$

and

$$I_q^{\gamma,-\gamma,\zeta}\{f(t)\} = I_q^{\gamma}\{f(t)\}, \tag{47}$$

then, Theorems 4 to 6 reduce to the following q-integral inequalities involving the Riemann-Liouville type of fractional q-integral operators.

Corollary 7. *Let f and g be two synchronous functions on* T*, and $h > 0$, then*

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{f(t)\} I_q^{\alpha}\{g(t)h(t)\} + I_q^{\alpha}\{g(t)\} I_q^{\alpha}\{f(t)h(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\alpha}\{f(t)g(t)\}, \tag{48}$$

for all $t > 0, 0 < q < 1$ and $\alpha > 0$.

Corollary 8. *Let f and g be two synchronous functions on* T*, and $h > 0$, then for* $t > 0$, $0 < q < 1$, $\alpha, \gamma > 0$,

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\gamma}\{f(t)g(t)h(t)\} + \frac{t^{\gamma}}{\Gamma_q(1+\gamma)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)h(t)\} + I_q^{\alpha}\{g(t)h(t)\} I_q^{\gamma}\{f(t)\} + I_q^{\alpha}\{g(t)\} I_q^{\gamma}\{f(t)h(t)\} + I_q^{\alpha}\{f(t)h(t)\} I_q^{\gamma}\{g(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\gamma}\{f(t)g(t)\} - I_q^{\alpha}\{f(t)g(t)\} I_q^{\gamma}\{h(t)\}. \tag{49}$$

Corollary 9. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality* (14), *then for all* $t > 0$, $\alpha, \gamma > 0, 0 < q < 1$,

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\gamma}\{f(t)g(t)h(t)\} - \frac{t^{\gamma}}{\Gamma_q(1+\gamma)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{g(t)h(t)\} I_q^{\gamma}\{f(t)\} - I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)h(t)\} + I_q^{\alpha}\{f(t)h(t)\} I_q^{\gamma}\{g(t)\} - I_q^{\alpha}\{g(t)\} I_q^{\gamma}\{f(t)h(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\gamma}\{f(t)g(t)\} + I_q^{\alpha}\{f(t)g(t)\} I_q^{\gamma}\{h(t)\}. \tag{50}$$

Special Cases

We now, briefly consider some of the consequences of the results derived in the previous sections. If we let $q \to 1^-$, and use the limit formulas:

$$\lim_{q\to 1^-} \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n \tag{51}$$

and

$$\lim_{q\to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha), \tag{52}$$

the results of Section 3 correspond to the results obtained in Section 2. Again, in view of the above limiting cases, Corollaries 8 and 9 provide, respectively, the q-extensions of the inequalities due to Sulaiman [8, pp. 24-25, Theorems 2.1 to 2.2].

Finally, if we consider the function h as constant > 0, the Theorems 1, 2, 4 and 5, and Corollaries 7 and 8 provide, respectively, the known results due to Purohit and Raina [6], and Öğünmez and Özkan [14].

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Una nueva clase de polinomios *q*-Apostol-Bernoulli de orden *α*

Mridula Garg and Subhash Alha

Department of Mathematics, University of Rajasthan, Jaipur 302004, Rajasthan, India
gargmridula@gmail.com and subhashalha@gmail.com

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Resumen

En este trabajo, en primer lugar se da una introducción de los números y polinomios de Bernoulli y sus *q*-generalizaciones. Luego se define una nueva clase de polinomios *q*-Apostol-Bernoulli de orden *α* y sus números correspondientes. Se obtienen representaciones explicitas, teorema de adición y fórmula diferenciales de esta nueva clase de polinomios.

Palabras clave: Polinomios *q*-Apostol-Bernoulli

A new class of *q*-Apostol-Bernoulli polynomials of order *α*

Abstract

In this paper, we first give an introduction of Bernoulli polynomials and numbers and their *q*-generalizations. We then define a new class of *q*-Apostol-Bernoulli polynomials of order *α* and corresponding numbers. We obtain explicit representations, addition theorem and differential formula for these newly defined class of polynomials.

Key words: *q*-Apostol-Bernoulli polynomials

Notations and Definitions

We shall use the following notations and definitions of *q*-theory (Gasper & Rahman [8])

The *q*-number $[x]_q$ and the *q*-number factorial $[n]_q!,\ n \in N$ are defined by

$$[x]_q = \frac{1-q^x}{1-q} \quad q \neq 1. \text{ and } [n]_q! = \prod_{j=1}^{n} [j]_q. \tag{1}$$

The *q*-shifted factorial (*q*-analogue of Pochhammer symbol) is defined as

$$(a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right),\ n \in N \text{ with } (a;q)_0 = 1,\ q \neq 1. \tag{2}$$

If we consider $(a;q)_\infty$ then as the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, therefore whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

Further, for any complex number α, we have

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{3}$$

There is one more definition of q-number shifted factorial,which is often used in the definitions of q-extension of Bernoulli polynomials. This is as follows

$$[a]_{q;n} = \prod_{j=0}^{n-1}[a+j]_q \text{, with } [a]_{q;n} = (1-q)^{-n}(q^a;q)_n. \tag{4}$$

The q-binomial theorem is given by

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad |z|<1,\ 0<|q|<1. \tag{5}$$

For $a = q^\alpha$; $(\alpha \in C)$ the result can be written as follows

$$\sum_{n=0}^{\infty}\frac{(q^\alpha;q)_n}{(q;q)_n}z^n = \frac{(q^\alpha z;q)_\infty}{(z;q)_\infty} = \frac{1}{(z;q)_\alpha} \qquad |z|<1,\ 0<|q|<1. \tag{6}$$

Introduction

The definitions of classical Bernoulli polynomials $B_n(x)$ and numbers B_n and their familiar generalizations $B_n^{(\alpha)}(x)$ and $B_n^{(\alpha)}$, Bernoulli polynomials and numbers of order α, can be seen in the texts (Erdélyi et al. [6]; Olver et al.[21]). Some interesting analogues of the classical Bernoulli polynomials were investigated by Apostol [1], so-called Apostol Bernoulli polynomials $B_n(x;\lambda)$. Further, Luo and Srivastava [18] introduced and investigated the Apostol-Bernoulli polynomials of order α, $B_n^{(\alpha)}(x;\lambda)$. Some more generalizations and analogues of these polynomials have been studied by researchers namely Natalini and Benardini[20], Luo et al.[14], Breeti[2], Srivastava et al. [23],Kurt [12], Tremblay [25].

q-analogues of Bernoulli numbers were first studied by Carlitz[3].Thereafter various other q-analogues of Bernoulli numbers and polynomials have been studied arising from varying motivations. Many authors have further studied and developed this subject, among which a few to mention are Koblitz [11], Tsumura [26], Srivastava et al. [24], Cenkciand Can[4],Ernst [7],Ryoo [22], Choi et al. [5], Kim et al. [10], Luo [17], Luo and Srivastava [15], Mahmudov [19] and Lee and Ryoo[13]. We recall here some of these definitions.

q-extensions of Bernoulli polynomials and numbers of order $\alpha \in C$, are defined by means of the following generating functions (see Luo and Srivastava [15])

$$(-z)^\alpha \sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^{n+x}e^{[n+x]_q z} = \sum_{n=0}^{\infty}B_{n;q}^{(\alpha)}(x)\frac{z^n}{n!}, \tag{7}$$

$$\left(-z\right)^{\alpha}\sum_{n=0}^{\infty}\frac{\left[\alpha\right]_{q;n}}{\left[n\right]_{q}!}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}B_{n;q}^{(\alpha)}\frac{z^{n}}{n!}. \tag{8}$$

The following formula for $B_{n;q}^{(\alpha)}\left(x\right)$ in terms of $B_{j;q}^{(\alpha)}$ can easily be obtained from (7) and (8)

$$B_{n;q}^{(\alpha)}\left(x\right)=\sum_{j=0}^{n}\binom{n}{j}\{[x]_{q}\}^{n-j}q^{(j+1-\alpha)x}B_{j,q}^{(\alpha)}. \tag{9}$$

Remark 1. *From the relation (9), it is obvious that the degree of* $B_{n,q}^{(\alpha)}\left(x\right)$ *is* $\left(n+1-\alpha\right)$ *in* q^{x}*, which means that for non integral values of* α*, it is not a polynomial in* q^{x}*.*

Cenkci and Can [4] introduced ***q***-extensions of Apostol-Bernoulli polynomials and numbers are defined by means of the following generating functions

$$\left(-z\right)\sum_{n=0}^{\infty}\lambda^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}\left(x;\lambda;q\right)\frac{z^{n}}{n!}, \tag{10}$$

$$\left(-z\right)\sum_{n=0}^{\infty}\lambda^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}\left(\lambda;q\right)\frac{z^{n}}{n!} \tag{11}$$

Choi et al. [5] gave the following definitions for q-extensions of Apostol-Bernoulli polynomials and numbers of order $k\in N$,

$$\left(-z\right)^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\lambda^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)\frac{z^{n}}{n!}, \tag{12}$$

$$\left(-z\right)^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\lambda^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}^{(k)}\left(\lambda;q\right)\frac{z^{n}}{n!}. \tag{13}$$

The following formula can easily be obtained from (12) and (13)

$$\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)=\sum_{j=0}^{n}\binom{n}{j}\{[x]_{q}\}^{n-j}q^{(j+1-k)x}\mathfrak{B}_{j}^{(k)}\left(\lambda;q\right) \tag{14}$$

Remark 2. *It is observed from (14), that the degree of* $\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)$ *is* $\left(n+1-k\right)$ *in* q^{x}*, whereas, the notation* $\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)$ *indicates that it should be of degree n.*

We would like to mention here that in the same paper the authors have also defined a q-extension of Apostol-Euler polynomials and numbers of order $k\in N$ by the following generating functions

$$2^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\left(-\lambda\right)^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathscr{E}_{n}^{(k)}\left(x;\lambda;q\right)\frac{z^{n}}{n!} \tag{15}$$

$$2^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\left(-\lambda\right)^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathscr{E}_{n}^{(k)}\left(\lambda;q\right)\frac{z^{n}}{n!} \tag{16}$$

From equations (3.18) and (3.32) of the same paper (Choi et al. [5]), it is easy to derive the following relation between $\mathfrak{B}_{n}^{(k)}\left(\lambda;q\right)$ and $\mathscr{E}_{n}^{(k)}\left(\lambda;q\right)$, which shows that these are not independent

$$\mathfrak{B}_n^{(k)}(-\lambda;q) = (-1)^n (k+1)_n \mathscr{E}_n^{(k)}(\lambda;q) \qquad (k \in N_0,\ n \in N) \tag{17}$$

In the present paper, we further extend this study and define a q-extension of Apostol-Bernoulli polynomials of order $B_{n,q}^{(\alpha,\lambda)}(x)$ $\alpha, \lambda \in C$. We shall also prove in Theorem 1 that these are polynomials of degree n in q^x. This property overcomes the shortcoming pointed out in Remark 2 of earlier definition $\mathfrak{B}_n^{(k)}(x;\lambda;q)$.

A new class of q-Apostol-Bernoulli polynomials and numbers of order α

Definition. For $\alpha, \lambda \in C$, $0 < |q| < 1$, we define a new class of q-Apostol-Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,\lambda)}(x)$ by means of the following generating function

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} \lambda^n q^n e^{[n+x]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!} \tag{18}$$

and corresponding numbers are given by

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} \lambda^n q^n e^{[n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)} \frac{z^n}{n!}. \tag{19}$$

Obviously, $B_{n,q}^{(\alpha,\lambda)} = B_{n,q}^{(\alpha,\lambda)}(0)$. (20)

Special Cases

1. If we set $\alpha = k \in N$ in (18) and (19), we get

$$(-1)^k \sum_{n=0}^{\infty} \frac{[k]_{q;n}}{[n]_q!} \lambda^n q^n e^{[n+x]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(k,\lambda)}(x) \frac{z^n}{n!}, \tag{21}$$

$$(-1)^k \sum_{n=0}^{\infty} \frac{[k]_{q;n}}{[n]_q!} \lambda^n q^n e^{[n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(k,\lambda)} \frac{z^n}{n!}. \tag{22}$$

We observe that $B_{n,q}^{(k,\lambda)}(x)$ is associated with $\mathfrak{B}_n^{(k)}(x;\lambda;q)$ given by (12) according with the following relation

$$B_{n,q}^{(k,\lambda)}(x) = \frac{n!}{q^x (n+k)!} \mathfrak{B}_{n+k}^{(k)}(x;\lambda;q). \tag{23}$$

Further on taking $k = 1$ in (21) and (22), we arrive at the following q-extension of Apostol-Bernoulli polynomials and numbers

$$(-1) \sum_{n=0}^{\infty} \lambda^n q^n e^{[n+x]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(1,\lambda)}(x) \frac{z^n}{n!}, \tag{24}$$

$$(-1) \sum_{n=0}^{\infty} \lambda^n q^n e^{[n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(1,\lambda)} \frac{z^n}{n!}. \tag{25}$$

Here, $B_{n,q}^{(1,\lambda)}(x)$ is associated with $B_n(x;\lambda;q)$ given by (10), according with the following relation

$$B_{n,q}^{(1,\lambda)}(x) = \frac{1}{q^x(n+1)} B_{n+1}(x;\lambda;q). \tag{26}$$

2. If we take $q \to 1$ in (18) and (19), we get the following class of Bernoulli polynomials $B_n^{(\alpha,\lambda)}(x)$ and numbers $B_n^{(\alpha,\lambda)}$

$$\frac{1}{(\lambda e^z - 1)^\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha,\lambda)}(x) \frac{z^n}{n!} \qquad (|z| < \ln\lambda) \tag{27}$$

$$\frac{1}{(\lambda e^z - 1)^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha,\lambda)} \frac{z^n}{n!} \qquad (|z| < \ln\lambda). \tag{28}$$

We observe that $B_n^{(\alpha,\lambda)}(x)$ is not comparable with the definition of Apostol-Bernoulli polynomials of order α, $B_n^{(\alpha)}(x;\lambda)$ defined by Luo and Srivastava [18] as follows

$$\left(\frac{z}{\lambda e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!}. \tag{29}$$

Rather it is related with Apostol Euler polynomials of order α, $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ [18] through the following relation

$$B_n^{(\alpha,-\lambda)}(x) = \frac{1}{(-2)^\alpha} \mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{30}$$

3. If we set $\lambda = 1$ in (18) and (19), we get the following q-extension of Bernoulli polynomials and numbers of order α

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} q^n e^{[n+x]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,1)}(x) \frac{z^n}{n!}, \tag{31}$$

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} q^n e^{[n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,1)} \frac{z^n}{n!}. \tag{32}$$

Here, $B_{n,q}^{(\alpha,1)}(x)$ is not comparable with the definitions $B_{n,q}^{(\alpha)}(x)$ given by (7) but we have the relation between $B_{n,q}^{(\alpha,1)}(x)$ and q-Euler polynomials of order α, $E_{n,q}^{(\alpha)}(x)$ (see [Luo and Srivastava [15]) by the following relation

$$B_{n,q}^{(\alpha,1)}(x) = \frac{(-1)^n}{(-2)^\alpha q^x} E_{n,q}^{(\alpha)}(x). \tag{33}$$

We would like to remark here that $B_{n,q}^{(\alpha,1)}(x)$ given by (31) has an improvement over $B_{n,q}^{(\alpha)}(x)$ given by (7) in the sense that for non integral values of α, it is polynomial of degree n in q^x as obvious from the relation (34).

Explicit Representations

Theorem 1. For $\alpha, \lambda \in C$, $0 < |q| < 1$, we have

(a) $$B_{n,q}^{(\alpha,\lambda)}(x) = \sum_{j=0}^{n} \binom{n}{j} \{[x]_q\}^{n-j} q^{jx} B_{j,q}^{(\alpha,\lambda)}, \tag{34}$$

(b) $$B_{n,q}^{(\alpha,\lambda)} = \frac{1}{(-1)^{\alpha}} \sum_{j=0}^{\infty} \frac{[\alpha]_{q;n}}{[j]_q!} q^j \lambda^j \left([j]_q\right)^n, \tag{35}$$

Where $B_{n,q}^{(\alpha,\lambda)}(x)$ and $B_{n,q}^{(\alpha,\lambda)}$ are defined by equation (18) and (19). It is clear from (34) that $B_{n,q}^{(\alpha,\lambda)}(x)$ is a polynomial of degree n in q^x.

Proof (a). Using the relation $[x+n]_q = [x]_q + q^x[n]_q$, we can write (18) as

$$\frac{e^{[x]_q z}}{(-1)^{\alpha}} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} \lambda^n q^n e^{q^x [n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{36}$$

Using (19) in L.H.S. of (36), we get

$$e^{[x]_q z} \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)} .q^{nx} . \frac{z^n}{n!} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{37}$$

Writing series for $e^{[x]_q z}$ we have the following

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)} .q^{nx} . \frac{\left([x]_q\right)^j}{j!} \frac{z^{n+j}}{n!} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{38}$$

Using series manipulation and equating coefficients of $\frac{z^n}{n!}$ we get the desired result (34).

(b). Result (35) can easily be obtained from (19) on using series of exponential function and equating coefficients of $\frac{z^n}{n!}$.

Form equation (34), it is easy to see that $B_{n,q}^{(\alpha,\lambda)}(x)$ is a polynomial of degree n in q^x.

We now calculate the values of q-Apostol-Bernoulli numbers $B_{n,q}^{(\alpha,\lambda)}$ and polynomials $B_{n,q}^{(\alpha,\lambda)}(x)$ for different values of n with the help of (35) and (34).

Few q-Apostol-Bernoulli numbers as calculated from (35) are

$$\left.\begin{aligned} B_{0,q}^{(\alpha,\lambda)} &= \frac{1}{(-1)^{\alpha}} \frac{1}{(q\lambda;q)_{\alpha}}, \\ B_{1,q}^{(\alpha,\lambda)} &= \frac{q\lambda}{(-1)^{\alpha}} \frac{[\alpha]_q}{(q\lambda;q)_{\alpha+1}}, \\ B_{2,q}^{(\alpha,\lambda)} &= \frac{q^2\lambda^2}{(-1)^{\alpha}} \frac{[\alpha]_q[\alpha+1]_q}{(q\lambda;q)_{\alpha+2}} + \frac{q\lambda}{(-1)^{\alpha}} \frac{[\alpha]_q}{(q^2\lambda;q)_{\alpha+1}}, \ldots \end{aligned}\right\} \tag{39}$$

Further using above values in (34), we get few q-Apostol-Bernoulli polynomials as follows

$$\left.\begin{aligned} B_{0,q}^{(\alpha,\lambda)}(x) &= B_{0,q}^{(\alpha,\lambda)} \\ B_{1,q}^{(\alpha,\lambda)}(x) &= \frac{1}{1-q} B_{0,q}^{(\alpha,\lambda)} - \frac{q^x}{1-q}\left[B_{0,q}^{(\alpha,\lambda)} - (1-q) B_{1,q}^{(\alpha,\lambda)}\right] \\ B_{1,q}^{(\alpha,\lambda)}(x) &= \frac{1}{1-q} B_{0,q}^{(\alpha,\lambda)} - \frac{2q^x}{1-q}\left[B_{0,q}^{(\alpha,\lambda)} - B_{1,q}^{(\alpha,\lambda)}\right] + \frac{q^{2x}}{1-q}\left[B_{0,q}^{(\alpha,\lambda)} - 2B_{1,q}^{(\alpha,\lambda)} + (1-q) B_{2,q}^{(\alpha,\lambda)}\right] \end{aligned}\right\} \tag{40}$$

Theorem 2. For $\alpha, \lambda \in C$, $0 < |q| < 1$, we have

$$B_{n,q}^{(\alpha,\lambda)}(x) = \frac{1}{(-1)^{\alpha}} \frac{1}{q^x} \Phi_{\alpha,q}(\lambda, -n, x). \tag{41}$$

where $\Phi_{\mu,q}(z,s,a)$ (see Choi et al.[5]) is a q-extension of generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$ defined by Goyal and Laddha[9] and is defined as follows

$$\Phi_{\mu,q}(z,s,a) = \sum_{n=0}^{\infty} \frac{[\mu]_{q;n}}{[n]_q!} \frac{q^{n+a}}{\left([n+a]_q\right)^s} z^n \qquad \left(\mu, s \in C; \operatorname{Re}(a) > 0\right). \tag{42}$$

Proof. In (18), if we write exponential function $e^{[n+x]_q z}$ in series form and compare it with (42) we easily arrive at (41).

If we let $q \to 1$ in (41), we get the following explicit representation for $B^{(\alpha\ \lambda)}(x)$

$$B_{n}^{(\alpha,\lambda)}(x) = \frac{1}{(-1)^{\alpha}} \Phi_{\alpha}^{*}(\lambda, -n, x) \qquad (|\lambda| < 1). \tag{43}$$

Addition Theorem

Theorem 3. For $\alpha, \lambda \in C$, $0 < |q| < 1$, we have

$$B_{n,q}^{(\alpha,\lambda)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \{[y]_q\}^{n-j} q^{jy} B_{j,q}^{(\alpha,\lambda)}(x) \tag{44}$$

Proof. It follows from (18) that

$$\frac{1}{(-1)^{\alpha}}\sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^n\lambda^n e^{[n+x+y]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{45}$$

Using the relation $[x+n+y]_q=[y]_q+q^y[x+n]_q$, (45) can be written as

$$\frac{e^{[y]_q z}}{(-1)^{\alpha}}\sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^n\lambda^n e^{q^y[n+x]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{46}$$

Using (18), the above result assumes the following form

$$e^{[y]_q z}\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x)q^{ny}\frac{z^n}{n!}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{47}$$

Expanding $e^{[y]_q z}$ in series form, using series manipulation and equating coefficients of $\frac{z^n}{n!}$, we get the result (44).

If we let $q\to 1$ in (44) we get the following addition formula for $B_n^{(\alpha,\lambda)}(x)$, defined by (27)

$$B_n^{(\alpha,\lambda)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha,\lambda)}(x)y^{n-k}. \tag{48}$$

If we take $\lambda=1$ in (44), we get the following result for q-extension of Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,1)}(x)$ defined by (31).

$$B_{n,q}^{(\alpha,1)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}\{[y]_q\}^{n-j}q^{jy}B_{j,q}^{(\alpha,1)}(x). \tag{49}$$

Differential Formula

Thoerem 4. For $\alpha, \lambda\in C$, $0<|q|<1$, we have

$$\frac{d}{dx}B_{n,q}^{(\alpha,\lambda)}(x)=n\frac{q^x\ln q}{(q-1)}B_{n-1,q}^{(\alpha,\lambda q)}(x). \tag{50}$$

Proof. Differentiating the generating function (18) w.r.t. x and using the following result

$$\frac{d}{dx}\left\{e^{q^n[x]_q z}\right\}=\frac{q^{n+x}\ln q}{q-1}ze^{q^n[x]_q z}, \tag{51}$$

we easily arrive at (50).

If we take the limit $q\to 1$ in (50), it gives the following differential formula for $B_n^{(\alpha,\lambda)}(x)$ defined by (27).

$$\frac{d}{dx}B_n^{(\alpha,\lambda)}(x)=nB_{n-1}^{(\alpha,\lambda)}(x). \tag{52}$$

If we take $\lambda = 1$ in (50), we get the following result for q-extension of Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,1)}(x)$ defined by (31).

$$\frac{d}{dx} B_{n,q}^{(\alpha,1)}(x) = n \frac{q^x \ln q}{(q-1)} B_{n-1,q}^{(\alpha,q)}(x). \tag{53}$$

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Algunos resultados que involucran operadores q-integrales fraccionales generalizados de Erdélyi-Kober

Leda Galué

(Dedicated to Professor Shyam Lal Kalla in occasion of his 76 years)

Universidad del Zulia. Facultad de Ingeniería
Centro de Investigación de Matemática Aplicada
Apartado de Correo 10482, e-mail: lgalue@hotmail.com
Maracaibo, Venezuela

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Resumen

En este trabajo se presentan algunos resultados para los operadores q-integrales fraccionales generalizados de Erdélyi-Kober definidos por Galué (2009). Además, se establecen desigualdades q-integrales fraccionales para funciones sincrónicas usando los operadores q-integrales fraccionales generalizados de Erdélyi-Kober antes mencionados. Algunos resultados dados por Belarbi y Dahmani, Öğünmez y Özkan, y Sulaiman se derivan como casos especiales de nuestros resultados.

Palabras clave: Operador q-integral fraccional generalizado de Erdélyi-Kober, desigualdades q-integrales fraccionales, q-integración fraccional por partes.

Some results involving generalized Erdélyi-Kober fractional q-integral operators

Abstract

In this paper some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué (2009) are presented. Also, fractional q-integral inequalities for synchronous functions are established using generalized Erdélyi-Kober fractional q-integral operators earlier mentioned. Some results due to Belarbi and Dahmani, Öğünmez and Özkan, and Sulaiman follow as special cases of our results.

Key words: Generalized Erdélyi-Kober fractional q-integral operator, fractional q-integral inequalities, fractional q-integration by parts.

Introduction

The most widely used definition of an integral of fractional order is via an integral transform, called the Riemann-Liouville operator of fractional integration: [1, p. 146]

$$\begin{aligned} {}_aI_x^{\alpha}\varphi(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt, \quad \mathrm{Re}(\alpha)>0, \\ &= \frac{d^n}{dx^n}\,{}_aI_x^{\alpha+n}\varphi(x), \qquad -n<\mathrm{Re}(\alpha)\le 0,\ n\in\mathbb{N}. \end{aligned} \tag{1}$$

Many authors, including Agarwal [2], Al-Salam [3], Kalia [4], Galué ([5]-[7]), Kalla *et al.* [8], Kalla and Kiryakova [9], Kiryakova [10], McBride and Roach [11], Ross [1], Saigo [12], Samko *et al.* [13], Saxena *et al.* [14], have defined and studied operators of fractional integration with their applications. We mention here some of these operators:

Erdélyi-Kober Operator: [3, p. 4, Eq. (20)]

$$\begin{aligned} I_{\eta,\alpha}f(x) &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt,\ \mathrm{Re}(\alpha)>0, \\ &= x^{-\alpha-\eta}\frac{d^n}{dx^n}\,x^{\eta+\alpha+n}I_{\eta,\alpha+n}f(x),\ -n<\mathrm{Re}(\alpha)\le 0. \end{aligned} \tag{2}$$

Basic analogue of Riemann-Liouville integral operator

Introduced by Al-Salam through [3]

$$I_q^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x \left(x-tq\right)_{\alpha-1}f(t)\,d_qt, \quad \alpha\in\mathrm{C},\ \mathrm{Re}(\alpha)>0. \tag{3}$$

q-analogue of Liouville fractional integral operator

The fractional q-integral operator $K_q^{-\alpha}$ is a q-analogue of Liouville fractional integral and it is defined by [3]

$$K_q^{-\alpha}f(x)=\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}\int_x^{\infty}\left(t-x\right)_{\alpha-1}f(q^{1-\alpha}t)\,d_qt, \quad \alpha\in\mathrm{C},\ \mathrm{Re}(\alpha)>0. \tag{4}$$

Basic analogue of Kober fractional integral operator

A basic analogue of Kober fractional integral operator has been defined by Agarwal [2] in the following form:

$$I_q^{\eta,\mu}f(x)=\frac{x^{-\eta-\mu}}{\Gamma_q(\mu)}\int_0^x t^{\eta}\left(x-tq\right)_{\mu-1}f(t)\,d_qt, \quad \eta,\mu\in\mathrm{C},\ \mathrm{Re}(\mu)>0, \tag{5}$$

where the order of integration μ is arbitrary real or complex number, and

$$(x-y)_{\nu}=x^{\nu}\prod_{n=0}^{\infty}\left[\frac{1-\left(y/x\right)q^{n}}{1-\left(y/x\right)q^{n+\nu}}\right]. \tag{6}$$

The result (5) can be expressed as [14]

$$I_q^{\eta,\mu} f(x) = \frac{(1-q)}{\Gamma_q(\mu)} \sum_{k=0}^{\infty} q^{k(1+\eta)} \left(1-q^{k+1}\right)_{\mu-1} f(xd^k). \tag{7}$$

Basic analogue of Weyl fractional integral operator

A basic analogue of Weyl fractional integral operator has been defined by Al-Salam [3] as follows:

$$K_q^{\eta,\mu} f(x) = \frac{q^{-\eta} x^{\eta}}{\Gamma_q(\mu)} \int_x^{\infty} (t-x)_{\mu-1} t^{-(\eta+\mu)} f(q^{(1-\mu)} t) d_q t, \quad \eta \in \mathrm{C},\ \mathrm{Re}(\mu) > 0. \tag{8}$$

The generalized Erdélyi-Kober fractional q-integral operator: Defined by [5]

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^x \left(x^{\beta} - t^{\beta} q\right)_{\mu-1} t^{\beta(\eta+1)-1} f(t)\, d_q t \tag{9}$$

$$= \beta(1-q^{1/\beta})(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^{\mu};q)_k}{(q;q)_k} q^{k(\eta+1)} f(xq^{k/\beta}). \tag{10}$$

$\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ \eta \in \mathrm{C}.$

As particular case of (9) we have

$$I_q^{0,\mu,1} f(x) = I_q^{0,\mu} f(x) = x^{-\mu} I_q^{\mu} f(x). \tag{11}$$

The generalized Weyl fractional q-integral operator: [6]

$$K_q^{\eta,\mu,\beta} f(x) = \frac{\beta q^{-\eta} x^{\beta\eta}}{\Gamma_q(\mu)} \int_x^{\infty} (t^{\beta} - x^{\beta})_{\mu-1} t^{-\beta(\eta+\mu-1)-1} f(q^{(1-\mu)/\beta} t) d_q t \tag{12}$$

$$= \beta(1-q^{1/\beta})(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^{\mu};q)_k}{(q;q)_k} q^{kn} f(xq^{-(\mu+k)/\beta}) \tag{13}$$

$$\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ \eta \in \mathrm{C}.$$

On the other hand, various researchers in the field of integral inequalities, motived by the usefulness of the fractional integral inequalities in fractional partial differential equations and in the solutions of fractional boundary value problems ([15]-[18]), have explored certain extensions and generalizations by involving fractional calculus operators. See for example references [15], [19]-[27].

In this paper some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué [5] are presented. Also, fractional q-integral inequalities for synchronous functions are established using generalized Erdélyi-Kober fractional q-integral operators earlier mentioned. Some results due to Belarbi and Dahmani [19], Öğünmez and Özkan [24], and Sulaiman [27] follow as special cases of our results.

Preliminares

In this section, we present some basic definitions, useful in our analysis.

The q-analogue of a complex number a is defined by [28]

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \in \mathbb{C} \setminus \{1\} \tag{14}$$

The q-shifted factorial is defined as [29]

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}), & n=1,2\ldots \\ [(1-aq^{-1})(1-aq^{-2})\ldots(1-aq^{-n})]^{-1}, & n=-1,-2\ldots \end{cases} \tag{15}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k), \tag{16}$$

which converges for $|q|<1$ and diverges for $a \neq 0$ and $|q|?1,$ and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in Z, |q|<1. \tag{17}$$

q-Gamma function

It is defined as follows: [29, p. 235, Eq. (I.35)]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{18}$$

Obviously,

$$\Gamma_q(x+1) = [x]_q\, \Gamma_q(x)$$

Basic hypergeometric series

This series is due to Heine (1846), [29]

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \tag{19}$$

where it is assumed that $c \neq q^{-m}$ for $m = 0,1,\ldots,$ and $(a;q)_n$ is the q-shifted factorial defined in (15).

The q-binomial theorem

One of the important summation formulae for hypergeometric series is given by the following binomial theorem:

$${}_2F_1(a,c;c;z) = {}_1F_0(a;-;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \quad |z|<1,$$

whose q-analogue was derived by Cauchy (1843), Heine (1847) and others [29]

$$ {}_1\phi_0(a;-;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;\,q)_\infty}, \quad |z|<1, |q|<1. \tag{20}$$

The q-derivative operator

This is denoted by D_q and defined for fixed q as [29, p. 22]

$$ \mathrm{D}_q f(z) = \frac{f(z)-f(qz)}{(1-q)\,z}, \quad z \neq 0, q \neq 1, \qquad \mathrm{D}_q f(0) = \lim_{z\to 0} \mathrm{D}_q f(z). \tag{21}$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathrm{R}$, we have linearity and product rules [30]

$$ \mathrm{D}_q\big(\alpha\, u(x) + \beta\, v(x)\big) = \alpha\big(\mathrm{D}_q u\big)(x) + \beta\big(\mathrm{D}_q v\big)(x) \tag{22}$$

$$ \mathrm{D}_q\big(u(x)\cdot v(x)\big) = u(qx)\big(\mathrm{D}_q v\big)(x) + v(x)\big(\mathrm{D}_q u\big)(x). \tag{23}$$

The q-integral

Thomae (1869) and Jackson (1910) introduced the q-integral in the following form [29, p. 19, Eqs. (1.11.2), (1.11.3)]

$$ \int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, $$

where

$$ \int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n. \tag{24}$$

The q-integration by parts

The following is the formula for the q-integration by parts [31]

$$ \int_a^b f(x)\big(\mathrm{D}_q g\big)(x)\, d_q x = \big[f(x)\, g(x)\big]_a^b - \int_a^b g(qx)\big(\mathrm{D}_q f\big)(x)\, d_q x. \tag{25}$$

Further Results For Generalized Erdélyi-Kober Fractional Q-integral Operators

In this section we establish some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué [5].

Theorem 1. For $\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ 0 < q < 1,$

i) If $\eta = 0,$ then

$$ I_q^{0,\mu,\beta} f(x) = \frac{\beta[1/\beta]_q}{[\mu]_q\,\Gamma_q(\mu)}\, f(0) + I_q^{0,\mu+1,\beta} \mathrm{D}_q f(x), \tag{26}$$

provided that $f(0)$ exists.

ii) If $\mathrm{Re}(\eta) > 0,$ then

$$I_q^{\eta,\mu,\beta} f(x) = q^\eta \, I_q^{\eta,\mu+1,\beta} \mathrm{D}_q\big(f(x)\big) + x^{-\beta}\,[\eta]_q I_q^{\eta-1,\mu+1,\beta} f(x) \tag{27}$$

Proof. Making in (9) a change of variable and using (21) and (14), we get

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta[1/\beta]_q \; x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^{x^\beta} \big(x^\beta - yq\big)_{\mu-1} \, y^\eta \, f(y^{1/\beta}) \, d_q y. \tag{28}$$

Since that [30]

$$\mathrm{D}_{q,t}(x-t)_\alpha = -[\alpha]_q \, (x-tq)_{\alpha-1}$$

we can write

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q \; x^{-\beta(\eta+\mu)}}{[\mu]_q \, \Gamma_q(\mu)} \int_0^{x^\beta} \mathrm{D}_{q,y}\big(x^\beta - y\big)_\mu \, y^\eta \, f(y^{1/\beta}) \, d_q y.$$

Now, using the q-integration by parts (25), we have

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q \; x^{-\beta(\eta+\mu)}}{[\mu]_q \, \Gamma_q(\mu)} \Big\{ \Big[\big(x^\beta - y\big)_\mu \, y^\eta \, f\big(y^{1/\beta}\big)\Big]_0^{x^\beta} -$$

$$\int_0^{x^\beta} \big(x^\beta - yq\big)_\mu \, \mathrm{D}_q\big(y^\eta \, f(y^{1/\beta})\big) \, d_q y \Big\}. \tag{29}$$

From this result making $\eta = 0,$ after of some calculations and using (28) we arrive to the result (26).

On the other hand, by applying the rule of the derivative of a product (23), (21) and (14)

$$\mathrm{D}_q\big(y^\eta \, f(y^{1/\beta})\big) = q^\eta \, y^\eta \, \mathrm{D}_q\big(f(y^{1/\beta})\big) + [\eta]_q \, y^{\eta-1} \, f(y^{1/\beta}) \tag{30}$$

therefore, from (29) and (30)

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q \; x^{-\beta(\eta+\mu)}}{[\mu]_q \, \Gamma_q(\mu)} \Big\{ \Big[\big(x^\beta - y\big)_\mu \, y^\eta \, f\big(y^{1/\beta}\big)\Big]_0^{x^\beta} -$$

$$q^\eta \int_0^{x^\beta} \big(x^\beta - yq\big)_\mu \, y^\eta \, \mathrm{D}_q\big(f(y^{1/\beta})\big) \, d_q y - [\eta]_q \int_0^{x^\beta} \big(x^\beta - yq\big)_\mu \, y^{\eta-1} \, f(y^{1/\beta}) \, d_q y \Big\}.$$

From this expression, after of some evaluations and using (28), we get the result (27).

Theorem 2. If $\mathrm{Re}(\beta) > 0, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\eta + 1 + \nu/\beta) > 0, \eta \in \mathbb{C}, 0 < q < 1$ then

$$I_q^{\eta,\mu,\beta}\{x^\nu\} = \beta\,[1/\beta]_q \, \frac{\Gamma_q(\eta+1+\nu/\beta)}{\Gamma_q(\mu+\eta+1+\nu/\beta)} x^\nu. \tag{31}$$

Proof. From (10) we get

$$I_q^{\eta,\mu,\beta}\left\{x^{\nu}\right\} = \beta(1-q^{1/\beta})(1-q)^{\mu-1}\; x^{\nu} \sum_{k=0}^{\infty} \frac{(q^{\mu};q)_k}{(q;q)_k} q^{k(\eta+1+\nu/\beta)}$$

now, using the q-binomial theorem given in (20) we have

$$\begin{aligned} I_q^{\eta,\mu,\beta}\left\{x^{\nu}\right\} &= \beta(1-q^{1/\beta})(1-q)^{\mu-1}\; x^{\nu}\; {}_1\phi_0\left(q^{\mu};-;q,q^{\eta+1+\nu/\beta}\right) \\ &= \beta(1-q^{1/\beta})(1-q)^{\mu-1} \frac{\left(q^{\mu+\eta+1+\nu/\beta};q\right)_{\infty}}{\left(q^{\eta+1+\nu/\beta};q\right)_{\infty}} x^{\nu}, \end{aligned}$$

finally, the application of (17) and (14) to this result leads us to (31).

Lemma 1. If K is a constant and $\mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0,\; \mathrm{Re}(\eta)>-1, 0<q<1,$ then

$$I_q^{\eta,\mu,\beta}\left\{K\right\} = \beta\,[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} K. \tag{32}$$

Proof. From (9) we obtain

$$I_q^{\eta,\mu,\beta}\left\{K\right\} = K\; I_q^{\eta,\mu,\beta}\left\{1\right\}$$

and using (31) with $\nu = 0$ we obtain of desired result.

Theorem 3. If $\mathrm{Re}(\beta), \mathrm{Re}(\mu)>0,\; \eta\in \mathbb{C}, 0<q<1$ then we have the following result for fractional q-integration by parts

$$\int_0^{\infty} g(q^{-\mu/\beta}x)\; I_q^{\eta-1+1/\beta,\mu,\beta} f(x)\; d_q x = \int_0^{\infty} f(x)\; K_q^{\eta,\mu,\beta} g(x)\; d_q x. \tag{33}$$

provided that the q-integrals exist.

Proof. Let be

$$I = \int_0^{\infty} g(q^{-\mu/\beta}x)\; I_q^{\eta-1+1/\beta,\mu,\beta} f(x)\; d_q x.$$

Using the definition of the operatos $I_q^{\eta,\mu,\beta}(.)$ given in (10) and interchanging the order of integration and summation, assuming absolute convergence, we get

$$I = \beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k} q^{k(\eta+1/\beta)} \int_0^{\infty} g(q^{-\mu/\beta}x) f\left(xq^{k/\beta}\right) d_q x,$$

now, making a simple variable change and using the result (21), we obtain

$$I = \beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k} q^{k\eta} \int_0^{\infty} g(q^{(-\mu-k)/\beta}w) f(w)\, d_q w,$$

and interchanging the order of summation and integration we have

$$I=\int_0^\infty f(w)\left[\beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1}\sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k}q^{k\eta}g(q^{(-\mu-k)/\beta}w)\right]d_q w.$$

Finally, interpreting this expression in terms of the operator $K_q^{\eta,\mu,\beta}(.)$ we obtain (33).

Particular cases: i) From (33) with $\beta = 1$:

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\eta,\mu} f(x)\, d_q x=\int_0^\infty f(x)\, K_q^{\eta,\mu} g(x)\, d_q x. \tag{34}$$

ii) Put $\eta = -\mu$ in (34) and using (3)-(4),

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\mu}(x^{-\mu} f(x)\ \ d_q x=q^{\mu(\mu+1)/2}\int_0^\infty x^{-\mu} f(x)\, K_q^{-\mu} g(x)\, d_q x,$$

that is,

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\mu} F(x)\, d_q x=q^{\mu(\mu+1)/2}\int_0^\infty F(x)\, K_q^{-\mu} g(x)\, d_q x, \tag{35}$$

which is a known result [32, p. 159, No. (5.25)].

Fractional q-integral inequalities

In this section, we establish some fractional q-integral inequalities employing generalized Erdélyi-Kober fractional q-integral operators defined in (9).

Definition. Two functions f and g are said to be synchronous on $[a, b]$, if

$$(f(x)-f(y))(g(x)-g(y)\ \geq 0, \tag{36}$$

for any $x, y \in [a, b]$.

Theorem 4. Let p be a positive function on $[0,\infty)$, f , g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\begin{aligned}\alpha[1/\alpha]_q\frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)}\, I_q^{\eta,\mu,\beta}\left(f(x)g(x)p(x)\right)+\beta[1/\beta]_q\frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu,\alpha}\left(f(x)g(x)p(x)\right)\geq\\ I_q^{\varepsilon,\nu,\alpha}\left(g(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)p(x)\right)+I_q^{\varepsilon,\nu,\alpha}\left(g(x)p(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)\right)+\\ I_q^{\varepsilon,\nu,\alpha}\left(f(x)\right)\, I_q^{\eta,\mu,\beta}\left(g(x)p(x)\right)+I_q^{\varepsilon,\nu,\alpha}\left(f(x)p(x)\right)\, I_q^{\eta,\mu,\beta}\left(g(x)\right)-\\ I_q^{\varepsilon,\nu,\alpha}\left(p(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)g(x)\right)-I_q^{\varepsilon,\nu,\alpha}\left(f(x)g(x)\right)\, I_q^{\eta,\mu,\beta}\left(p(x)\right)\end{aligned} \tag{37}$$

where $x>0,\ 0<q<1,\ \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\eta)>-1, \mathrm{Re}(\varepsilon)>-1.$

Proof. Since the functions *f* and *g* are synchronous functions on $[0,\infty)$ therefore from definition (36), we have

$$\left(f(t)-f(y)\right)\left(g(t)-g(y)\right)\geq 0, \qquad t, y\in[0,\infty)$$

then as p is a positive function on $[0,\infty)$

$$\big(f(t)-f(y)\big)\big(g(t)-g(y)\big)\big(p(t)+p(y)\big)\geq 0,$$

that is,

$$f(t)g(t)p(t)+f(t)g(t)p(y)+f(y)g(y)p(t)+f(y)g(y)p(y)\geq$$

$$f(t)g(y)p(t)+f(t)g(y)p(y)+f(y)g(t)p(t)+f(y)g(t)p(y) \tag{38}$$

Let be

$$F(\eta,\mu,\beta;x,t)=\frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)}\left(x^{\beta}-t^{\beta}q\right)_{\mu-1}t^{\beta(\eta+1)-1}\quad x>0,t\in(0,x) \tag{39}$$

Observe from (6) that

$$F(\eta,\mu,\beta;x,t)=\frac{\beta}{\Gamma_q(\mu)}\,x^{-\beta(\eta+1)}\,t^{\beta(\eta+1)-1}\qquad\times$$

$$\frac{\left(1-\frac{t^{\beta}}{x^{\beta}}q\right)\left(1-\frac{t^{\beta}}{x^{\beta}}q^{2}\right)\cdots\left(1-\frac{t^{\beta}}{x^{\beta}}q^{k+1}\right)\cdots}{\left(1-\frac{t^{\beta}}{x^{\beta}}q^{\mu}\right)\left(1-\frac{t^{\beta}}{x^{\beta}}q^{\mu+1}\right)\cdots\left(1-\frac{t^{\beta}}{x^{\beta}}q^{\mu+k}\right)\cdots},\quad k=0,1,2,\cdots$$

then $F(\eta,\mu,\beta;x,t)$ is always positive for all $x>0,t\in(0,x)$, since that $\left(1-\frac{t^{\beta}}{x^{\beta}}q^{\lambda}\right)>0,$ for $\mathrm{Re}(\beta)>0,\ 0<q<1,\mathrm{Re}(\lambda)>0,$ and the other terms are also positive under the conditions established in the theorem.

Multiplying (38) by $F(\eta,\mu,\beta;x,t),$ taking the q-integration from the result with respect to t from 0 to x, and keeping in mind the definition of $I_q^{\eta,\mu,\beta}(\cdot)$ operator, we get

$$I_q^{\eta,\mu,\beta}(f(x)g(x)p(x)\ +p(y)I_q^{\eta,\mu,\beta}(f(x)g(x)\ +f(y)g(y)I_q^{\eta,\mu,\beta}(p(x)\ +$$

$$\beta[1/\beta]_q\frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)}f(y)g(y)p(y)\geq g(y)I_q^{\eta,\mu,\beta}(f(x)p(x)\ +$$

$$g(y)p(y)I_q^{\eta,\mu,\beta}(f(x)\ +f(y)I_q^{\eta,\mu,\beta}(g(x)p(x)\ +f(y)p(y)I_q^{\eta,\mu,\beta}(g(x)), \tag{40}$$

where we use (32).

Similarly, multiplying (40) by $F(\varepsilon,\nu,\alpha;x,y),$ taking the q-integration from the result with respect to y from 0 to x, and keeping in mind the definition of $I_q^{\varepsilon,\nu,\alpha}(\cdot)$ operator, we get (37).

If in the Theorem 4 we put $\varepsilon=\eta,\ \nu=\mu,\ \alpha=\beta$ we get:

Corollary 1. Let p be a positive function on $[0,\infty)$ f,g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\beta[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\eta,\mu,\beta}\big(f(x)g(x)p(x)\big) \geq I_q^{\eta,\mu,\beta}\big(g(x)\big)\ I_q^{\eta,\mu,\beta}\big(f(x)p(x)\big)+$$

$$I_q^{\eta,\mu,\beta}\big(g(x)p(x)\big)\ I_q^{\eta,\mu,\beta}\big(f(x)\big) - I_q^{\eta,\mu,\beta}\big(p(x)\big)\ I_q^{\eta,\mu,\beta}\big(f(x)g(x)\big), \tag{41}$$

where $x>0,\ 0<q<1,\ \mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0, \mathrm{Re}(\eta)>-1$.

If $p(x)=x^{\lambda}$ in the Theorem 4 we have:

Corollary 2. Let f and g be synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\alpha[1/\alpha]_q \frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)} I_q^{\eta,\mu,\beta}\big(x^{\lambda} f(x)g(x)\big) + \beta[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu,\alpha}\big(x^{\lambda} f(x)g(x)\big) \geq$$

$$I_q^{\varepsilon,\nu,\alpha}\big(g(x)\big)\ I_q^{\eta,\mu,\beta}\big(x^{\lambda} f(x)\big) + I_q^{\varepsilon,\nu,\alpha}\big(x^{\lambda} g(x)\big)\ I_q^{\eta,\mu,\beta}\big(f(x)\big) + I_q^{\varepsilon,\nu,\alpha}\big(f(x)\big)\ I_q^{\eta,\mu,\beta}\big(x^{\lambda} g(x)\big)+$$

$$I_q^{\varepsilon,\nu,\alpha}\big(x^{\lambda} f(x)\big)\ I_q^{\eta,\mu,\beta}\big(g(x)\big) - \alpha[1/\alpha]_q \frac{\Gamma_q(\varepsilon+1+\lambda/\alpha)}{\Gamma_q(\nu+\varepsilon+1+\lambda/\alpha)} x^{\lambda}\ I_q^{\eta,\mu,\beta}\big(f(x)g(x)\big) -$$

$$\beta[1/\beta]_q \frac{\Gamma_q(\eta+1+\lambda/\beta)}{\Gamma_q(\mu+\eta+1+\lambda/\beta)} x^{\lambda}\ I_q^{\varepsilon,\nu,\alpha}\big(f(x)g(x)\big) \tag{42}$$

where $x>0,\ 0<q<1,\ \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\eta+1+\lambda/\beta)>0,$ $\mathrm{Re}(\varepsilon+1+\lambda/\alpha)>0, \mathrm{Re}(\eta)>-1, \mathrm{Re}(\varepsilon)>-1.$

Proof. This result is obtained directly from (37) by replacing $p(x)$ by x^{λ} and using (31).

Now, if we make $\alpha=\beta=1$ then the Theorem 4 reduces to the following q-integral inequality involving basic analogue of Kober fractional integral operator:

Corollary 3. Let p be a positive function on $[0,\infty)$, f, g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu}(\cdot)$ a fractional q-integral operator, as defined by (5), then

$$\frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)} I_q^{\eta,\mu}\big(f(x)g(x)p(x)\big) + \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu}\big(f(x)g(x)p(x)\big) \geq$$

$$I_q^{\varepsilon,\nu}\big(g(x)\big)\ I_q^{\eta,\mu}\big(f(x)p(x)\big) + I_q^{\varepsilon,\nu}\big(g(x)p(x)\big)\ I_q^{\eta,\mu}\big(f(x)\big)+$$

$$I_q^{\varepsilon,\nu}\big(f(x)\big)\ I_q^{\eta,\mu}\big(g(x)p(x)\big) + I_q^{\varepsilon,\nu}\big(f(x)p(x)\big)\ I_q^{\eta,\mu}\big(g(x)\big)-$$

$$I_q^{\varepsilon,\nu}\big(p(x)\big)\ I_q^{\eta,\mu}\big(f(x)g(x)\big) - I_q^{\varepsilon,\nu}\big(f(x)g(x)\big)\ I_q^{\eta,\mu}\big(p(x)\big) \tag{43}$$

where $x>0,\ 0<q<1, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\eta)>-1, \mathrm{Re}(\varepsilon)>-1.$

Special Cases: i) Taking $\lim\limits_{q\to 1^-}$ in Theorem 4 and Corollaries 1-3, and additionally making $\alpha=\beta=1$ in Theorem 4 and Corollaries 1-2, we obtain integral inequalities involving Erdélyi-Kober operators.

ii) For $\lambda=0,\eta=\varepsilon=0,\alpha=\beta=1$ in Corollary 2 and using (11) we get,

$$\frac{x^{\nu}}{\Gamma_q(\nu+1)}I_q^{\mu}\big(f(x)g(x)\big)+\frac{x^{\mu}}{\Gamma_q(\mu+1)}I_q^{\nu}\big(f(x)g(x)\big)\geq$$

$$I_q^{\nu}\big(g(x)\big)\,I_q^{\mu}\big(f(x)\big)+I_q^{\nu}\big(f(x)\big)\,I_q^{\mu}\big(g(x)\big) \tag{44}$$

with $x>0,\ 0<q<1,\mathrm{Re}(\mu)>0,\mathrm{Re}(\nu)>0.$

This result corresponds to quantun version of the same given by Öğünmez and Özkan [24, p. 5, No. (3.11)].

Taking $\lim\limits_{q\to 1^-}$ in (44) we obtain the result given by Belarbi and Dahmani [19, p. 188, No. (16)].

iii) For $\eta=\varepsilon=0$ in Corollary 3 and applying (11) we have,

$$\frac{x^{\nu}}{\Gamma_q(\nu+1)}I_q^{\mu}\big(f(x)g(x)p(x)\big)+\frac{x^{\mu}}{\Gamma_q(\mu+1)}I_q^{\nu}\big(f(x)g(x)p(x)\big)\geq$$

$$I_q^{\nu}\big(g(x)\big)\,I_q^{\mu}\big(f(x)p(x)\big)+I_q^{\nu}\big(g(x)p(x)\big)\,I_q^{\mu}\big(f(x)\big)+$$

$$I_q^{\nu}\big(f(x)\big)\,I_q^{\mu}\big(g(x)p(x)\big)+I_q^{\nu}\big(f(x)p(x)\big)\,I_q^{\mu}\big(g(x)\big)-$$

$$I_q^{\nu}\big(p(x)\big)\,I_q^{\mu}\big(f(x)g(x)\big)-I_q^{\nu}\big(f(x)g(x)\big)\,I_q^{\mu}\big(p(x)\big) \tag{45}$$

$x>0,\ 0<q<1,\mathrm{Re}(\mu)>0,\mathrm{Re}(\nu)>0,$

which is a known result [27, p. 456, No. (3.2)].

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Revisión de la teoría de Boehmians

P. K. Banerji and Deshna Loonker

(Dedicated to Professor Shyam Lal Kalla in occasion of his 76 years)

Department of Mathematics
Faculty of Science, J. N. V. University
JODHPUR - 342 005, INDIA
banerjipk@yahoo.com; deshnap@yahoo.com

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Resumen

Este artículo está organizado en dos secciones seguidas de una lista de artículos seleccionados de los autores. Probablemente este es el primer intento de escribir con todo detalle acerca de los operadores de Boehmians y J. Mikusinski junto con las contribuciones de varios matemáticos respecto a Bohemians. La teoría de distribuciones de Schwartz fue desarrollada para dar un soporte y fundamentos matematicos comprensibles en la generalización de las propiedades de la función delta de Dirac. A partir del trabajo de Sobolev y Schwartz, se hicieron intentos para generalizar el concepto de distribuciones. Colombeau construyó una nueva algebra diferenciable de funciones generalizadas conteniendo el espacio de distribución, en el cual el producto puede ser definido. El concepto de Boehmians, la mas reciente generalizacion de la teoria de distribuciones de Schwartz, esta motivada por los operadores regulares introducidos y por Boehme. Boehme no adoptó el nombre de Teoria de Bohemians, sino J. Mikusinski y P. Mikusiński fueron inspirados para desarrolar la Teoria de Boehme, que posiblemente adopto el nombre de Boehmians.

Palabras clave: Calculo operacional, operaciones de Mikusinski, funciones generalizadas, distriubuciones de Schwartz, aproximaciones secuencial y funcional, Boehmian, espacio de Beohmian, Boehmian inteprable, boehmian ajustado, ultra Boehmian

Boehmians revisited

Abstract

This article is organized in two sections followed by a list of selected research articles of the authors. Presumably this is the first attempt to write every major and minor detials about Boehmians and J. Mikusinski operators under one cover together with major contributions of various mathematicians with regard to Boehmians. The theory of Schwartz distributions was developed in order to give a concerte and comprehensible mathematical foundation for generalzing the properties of Dirac $\delta -$ function. Starting from the work of Sobolev and Schwartz, attempts were made to generalize the concept of distributions. Colombeau constructed a new differentiable algebra of generalized functions containing the space of distribution, in which product can be defined. The concept of Boehmians, one of the youngest generalization of Schwartz theory of distributions, is motivated by the regular operators introduced by Boehme. Boehme did not, himself, coined the name, the Boehmian, rather J. Mikusiński and P. Mikusiński were inspired to develop the theory of Boehme, which possibly (the conjecture) coined the name Boehmian.

Key words: Operational calculus, Mikusiński operators, generalized functions, Schwartz distributions, sequential and functional approaches, Boehmian, Boehmian space, integrable Boehmian, tempered Boehmians, ultraBoehmian.

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On Operational Calculus and Mikusiński Operators

Theory of operators is an apparent prerequisite to study Boehmians and that brings the concept of operational calculus closer to build a better understanding. Plesner [1], while studying the spectral theory of linear operators, reinforced the foundation of operational calculus, which was later extended by Detkin [2]. In the general theory of linear operators, function-operators play an important role. A rule of correspondence established between a set of functions and a class of operators is : *To every function* $F(\lambda)$ *of a given set of functions there corresponds a unit operator* $F(A)$ *and to the unit function* $F(\lambda)=1$*, there corresponds a unit operator* E *and to the function* $F(A)=\lambda$ *an operator* A. Matter of fact, the question relates to the isomorphism between classes of operators and classes of functions, with a unit operator corresponding to a unit function, and the operator A to the function $F(\lambda)=\lambda$, whereas, to the sum and product of functions, $F_1(\lambda)+F_2(\lambda)$ and $F_1(\lambda)F_2(\lambda)$, there correspond the sum and product of corresponding operators.

The use of Laplace transforms restricted the range of applicability of operational calculus techniques, which initiated Jan Mikusiński choose to revert to the original operational view point that did not depend on the Laplace transform. Having started from

$$fg=\int_0^t f(t-\tau)g(\tau)d\tau$$

like Heavisides, he obtained an operational calculus through a straight forward algebraic path. Mikusiński begun from the algebra of functions, where the convolution played the role of product. Even Mikusiński's operational calculus underwent remarks of containing deficiencies because of outright rejection of the Laplace transform which obstructs the realization of some operational formulae. Raevskii could, later, circumvent the difficulty by replacing Mikusiński's expression by a convenient expression

$$fg=\frac{d}{dt}\int_0^t f(t-\tau)g(\tau)d\tau.$$

The nucleus of Mikusiński's reasoning is the idea of the operators, named after him, the theory of which was established during 1950-52. He has represented the genus of fractional number of the type f/g, where f and g are functions in the limit $0\le x<\infty$. The division (f/g) is understood as an operation, which is the inverse of convolution. If the convolution of two functions f and h is denoted by $f*h$, then $h=f/g$. Polish and German scholars have extended Mikusiński's perceptions. Mikusiński had considered his operators a primitive on an infinite interval. Passage of time in fifties developed the theory of operators on a finite interval, based on the preceding theory.

The algebraic treatment of Mikusiński's operational calculus widened the scope of applications of the techniques of which, according to him, "if the class of functions for which the Laplace transform exists, then two approaches, one due to Mikusiński and the other the Laplace transform technique, are equivalent. However, in the class of functions defined in a finite interval the Laplace or, for that matter, any other transformation, does not reduce a transcendental problem to an algebraic one". Infact, any transformation can not translate the convolution

$$\int_0^t f(t-\tau)g(\tau)d\tau \quad ,$$

with $f(t)=0$, in the first half of the given interval to the usual product, since this convolution equals zero [cf. Mikusiński [3], Shtokalo [4] for more details].

Unlike Mikusiński's first theory of operational techniques (1950-52), which is dealt with (briefly) in the preceding section, the theory propounded here is algebraic in nature and considered as an alternate approach to the problem of constructing a consistent theory of generalized functions [Mikusiński [5]]. It projects the process by which the concept of number is extended from integers to rational numbers and provides a natural approach to operational calculus as well as to generalized functions. Although it (Mikusiński's theory) is successful with functions defined on the positive real line and has been extended to functions of several variables of such type, yet it is not suitable to deal with functions of unrestricted real variable or with functions on an arbitrary region of a space of n dimensions $(n \geq 2)$.

Mikusiński showed that the set $C[0,\infty)$, the letter C suggests continuous, with addition and multiplication by scalars defined in an obvious way and multiplication of two functions a,b of the set defined by the convolution $a*b$, forms a commutative ring, which is called the convolution ring. By virtue of Titchmarsh's theorem [Sneddon [6], page 68], we observe that in this convolution ring, division is a meaningful operation. Familiarity of this is found in when idea of division of integers is dealt with, where division by extending the concept of number from integers to rational numbers in terms of classes of equivalent ordered pairs of integers is ensured. In the present context, we consider ordered pairs of elements of $C[0,\infty)$ and consider (a,b) and (c,d) to be equivalent, if $a*d = c*b$. The class of all ordered pairs of continuous functions, equivalent to (a,b), is denoted by a/b, which is called a convolution quotient.

In C, the set of all convolution quotient, we can define the operations of addition, multiplication by a scalar, and multiplication and show that embedding of $C[0,\infty)$ in C preserves all these operations. That allows, therefore, to write $(a*f)/a$ as f for any pair of continuous functions a and f and $(\lambda a)/a$ as λ for any scalar λ. The unit element e in C may be written as a/a and it can be shown that multiplication by e reproduces f, which confirms the identification of unit element in C with the Dirac delta function.

The assumption is that the positive real axis is considered ($t \geq 0$). Construction of the rational numbers from the integers is mandatory to know Mikusiński operators and later, the Boehmians; but those who are familiar with this may omit this part. The technicality involves the establishment of the equivalence relations for the ordered pair of integers, the separation into equivalence classes, and the verification of the independence of choice of representative of an equivalence class, for instance 6/5 and 30/25 are both representative for the class of quotients which are equivalent to 6/5. Now care must be taken to construct the field of quotients, the naming of equivalence classes as rational numbers, and embedding of the integers into this new system. We write a/b (for notations) to represent the elements of the new system, where a and b are integers. The second element is not always zero. The operations of addition and multiplication is simple matter for that cause. We observe that b/b plays the role of unit element in the new system and that $0/b$ plays the role of zero. If $a \neq 0$, then the equation $(a/b)\,(x/y) = (c/d)$ has the solution $(bc)/(ad)$ and it is unique. All these and little more make available the operational calculus in which division is possible. In reference to what is written above, we obtain a model for the construction of the convolution quotient from the continuous functions by the construction of the rational numbers form the integers. Set of functions, which are continuous for $t \geq 0$ are considered, the addition that is taken is pointwise. Since pointwise multiplication does have non-trivial divisors of zero, whereas convolution does not have so, the multiplication considered is convolution. We point out that equivalence class (mentioned above) is named as Mikusiński operators or as generalized functions.

For $g \neq \{0\}$, which is the constant function, the convolution quotient g/g must appear, which leads us to a unit element among the generalized functions. It may be noted that the subset of the Mikusiński operator satisfies all of the properties of real (complex) numbers as well as the operational properties of continuous functions, e.g. addition, multiplication by real (complex) numbers and convolution, which are required for multiplication of functions by numbers. As a consequence, within the field of Mikusiński operators (generalized functions) we can now consider addition of a number and a function, see Buschman [7] for relevant information.

We should have complied with the regulation of describing the generalized functions or the Schwartz theory of distributions, of which the Boehmian is called the youngest generalization, prior talking about Boehmians. We have, at the last moment, negated this idea for the simple fact that generalized functions have wide familiarity among readers, and moreover, due to paucity of space. However, we give a brief note on the generalized functions prior moving to the section(s) only on Boehmians.

There are some problems encountered in applied mathematics when transform methods are applied to analyze physical situations in which impulsive forces or point sources are involved. Introduction of Dirac delta function simplifies formal calculations. But the rules for doing the manipulation do not follow, in natural way, the methods of classical analysis. This, possibly, led to the advent of the concept of generalized functions. Bochner [8] and Sobolev [9] have coined first ideas of such an approach but the firm foundation was put by the work of Schwartz [10], culminating in the publication of his treatise. Zemanian [11, 12] exhibit excellent study on this concept.

Paul Dirac [13] introduced, for the first time, in quantum mechanical studies, the delta function which possesses the property $\delta(x) = 0, x \neq 0$ and $\delta(x)\varphi(x)dx = \varphi(0), \varphi \in C$. It was soon pointed out by mathematicians that from purely mathematical point of view this definition is meaningless. It was, of course, even clear to Dirac himself that the δ - function is not a function in the classical sense and, what is important, it operates as an operator (more precisely as a functional) that relates, via above formula to each continuous function φ a number $\varphi(0)$, which is its value at a point O.

The simplest attempt at such a generalization,i.e., to generalize the entire concept of a function, is due to Mikusiński [14] which is developed by Temple [15, 16]. This method defines generalized functions as classes of equivalent fundamental sequences of continuous functions, which is similar to that used when real numbers are introduced with the help of fundamental sequences of rational numbers. References for further reading, among many others, are Beltrami and Wohlers [17], Bremermann [18], Carmichael and Pilipovic [19], Debnath [20], Debnath and Mikusiński [21], Erdélyi [22], Friedmann [23], Gel'fand and Shilov [24, 25, 26, 27, 4 vols.], Hoskins [28], Korevarr [29], Mikusiński and Sirorski [30], Pandey [31], Zemanian [11,12]. Distributions are generalization of locally integrable functions on the real line, or more generally a generalization of functions which are defined on an arbitrary open set in the Euclidean space. The mathematical theory called the theory of distributions, which enabled the introduction of the Dirac delta function without any logical restrictions, was coined in forties of the preceding century. As once the theory of real number was generalized, this theory generalized the notion of function.

The two most important approaches in theory and practice are: functional approach advented by Soboleff [9] and Schwartz [32] where distributions are defined as linear functionals continuous in linear spaces; sequential approach given by Mikusiński [14] , where distributions are defined as class of equivalent sequences. It may be noted that among important, in practice operations, are regular and non-regular operations. For example, the two argument operations of product $A(\varphi,\psi) = \varphi \cdot \psi$ and the convolution $A(\varphi,\psi) = \varphi * \psi$ are not regular operations and, therefore, they cannot be defined for arbitrary distributions. Mikusiński [33, 34] devised a general method to define irregular operations on distributions (see also Antosik et al. [35]).One may also refer to Mikusiński [36], Kaminski [37], Antosik and Ligeza [38] among others.

The alternate approach is to study distributions as limit of sequences of functions. Logical construction of such limits is based on the Cantor's concept of equivalence classes. To each distribution in the functional approach there corresponds one distribution in the sequential approach, and conversely. The approach is to establish, first, the usual property of the distribution as a derivative of continuous function and then develop the remaining on the basis of both, as a limit of continuous functions and as a derivative of a function of a distribution [cf. Lojasiewicz [39] and Zielenzny [40]].

The fundamental term to form the basis of the sequential approach is the identification principle. Oriented segments x and y are said to be equivalent if they are parallel and have the same length and orientation, we write $x : y$, which has the properties, (i) $x \sim x$ (reflexive), (ii) if $x \sim y$, then $y \sim x$ (symmetric) and (iii) if $x \sim y$, $y \sim z$, then $x \sim z$ (transitive). By means of equivalence relation we obtain a decomposition of the set of all oriented segments into disjoint classes such that the segments in the same class are equivalent and in different classes they are not, by virtue of Cantor's definition of real numbers. Rational numbers are the basic concepts for understanding Cantor's theory, the functions continuous in $A < x < B$ $(-\infty \leq A < B \leq \infty)$ are the starting point for the theory of sequential approach. A sequence $\{F_n(x)\}$ of continuous functions $(A < x < B)$ is called a fundamental sequence if there exists a sequence $\{F_n(x)\}$ and an integer $k \geq 0$ such that $F_n^{(k)} = f_n(x)$ and the sequence $\{F_n(x)\}$ converges almost uniformly.

If there exists sequences $\{F_n(x)\}$ and $\{G_n(x)\}$ and an integer $k \geq 0$ such that

(i) $F_n^{(k)} = f_n(x)$ and $G_n^{(k)} = g_n(x)$

(ii) $F_n(x) \Rightarrow \Leftarrow G_n(x)$,

then the fundamental sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are said to be equivalent, we write $\{f_n(x)\} \sim \{g_n(x)\}$. In other words, fundamental sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are equivalent if and only if the sequence given by $f_1(x), g_1(x), f_2(x), g_2(x), \ldots$ is fundamental and moreover, in that case there exists an integer $k \geq 0$ and the continuous functions $F_n(x)$ and $G_n(x)$ such that $F_n^{(k)} = f_n(x)$, $G_n^{(k)} = g_n(x)$ and the sequence $F_1(x), G_1(x), F_2(x), G_2(x), \ldots$ converges uniformly and, consequently, (i) and (ii) hold true. By virtue of conditions (i) - (iii) (i.e. reflexive, symmetric and transitive as dfined above), the set of all fundamental sequences $\{f_n(x)\}, A < x < B$, is partitioned into equivalence classes without common elements such that two fundamental sequences are in the same equivalence class if and only if they are equivalent, which (the equivalence classes) will be called distribution in $A < x < B$. The notion of the distributions is, thus, obtained, from the identification of equivalent fundamental sequences, those distributions are denoted by $[f_n(x)]$.

Denoting by 0, is the zero distribution, which is the distribution coinciding with the function identically equal to zero, we mean $0 + f(x) = f(x)$ and $0 \cdot f(x) = 0$. The symbol 0 has two interpretations, for the former it means number zero and for the latter, it is zero distribution, Loonker [41] and Krystyna [42]. A formal definition of distribution was due to Mikusiński [43], based on which Mikusiński and Sikorski [30, 44] developed sequential theory of distributions and later Mikusiński and Antosik [35] wrote the monograph. Mikusiński's definition of distributions in sequential sense is analogue to the definition of real numbers in the Cantor's theory.

The operations, addition of smooth functions, difference of smooth functions, multiplication of a smooth function by a fixed number $\lambda;\lambda\varphi$, translation of the argument of a smooth function $\varphi(x+h)$, derivation of a smooth function of a fixed order m; $\varphi^{(m)}$, multiplication of a smooth function by a smooth fixed function ω; $\omega\varphi$, substitution of a fixed smooth function $\omega \neq 0$, product of a smooth function with separated variables; $\varphi_1(x)\varphi_2(y)$ convolution of a smooth function with a fixed function ω from the space D (of smooth functions whose supports are bounded); $(f \bullet \omega)(x) = \int_{\mathrm{R}} \varphi(x-t)\omega(t)dt$, inner product of a smooth function with a fixed function form the space Δ; $(\varphi, \omega) = \int_{\mathrm{R}} \varphi(x)\omega(x)dx$, are all regular. An advantage of the sequential approach to the theory of distributions is the simplified way of extending to distributions many operations which are regular. Moreover, we know that every distribution is locally a distributional derivative of a finite order of a continuous function. It is also a natural consequence that the sequence $(f \bullet \delta_n)$ is distributionally convergent to f (i.e., fundamental for f) for an arbitrary distribution $f \in \mathrm{R}^1$, where (δ_n) is the delta sequence.

Jan Mikusiński also explored the theory of integration. His definition of the Lebesgue integral is simple and possesses clean geometrical meaning, which can be formulated for functions defined in

R^k with values in a Banach space which yields a uniform approach to the Lebesgue and the Bochner integrals (both have a great cohesiveness to Boehmians). The function $f:\mathsf{R}^k \to X$ $(f:\mathsf{R}^k \to \mathsf{R}^1)$, where X is a Banach space, is called Bochner (Lebesgue) integrable if there exists a sequence of interval $I_n = [a_{1n}, b_{1n}) \times \ldots \times a_{kn}, b_{kn})$ in R^k in R^k and a sequence $(\lambda_n)_{n\in N}$ of elements of X such that

$$\sum_{n=1}^{\infty} |\lambda_n| \operatorname{vol}(I_n) < \infty$$

and

$$f(x) = \sum_{n=1}^{\infty} \lambda_n \chi I_n(x),$$

at those points x at which the series is absolutely convergent, where χ_I denotes the characteristic function of an interval I.

Bochner (Lebesgue) integral of a function satisfying above conditions, is defined by

$$\int f = \sum_{n=1}^{\infty} |\lambda_n| \operatorname{vol}(I_n),$$

which is equivalent to the classical definitions of Bochner and Lebesgue integrals, see Mikusiński [45].

Introducing Boehmians

That we are writing in this section is one of the youngest generalizations of functions and more particularly that of Schwartz theory of Distributions, devised by Thomas Kalman Boehme, descendant of Prof. Arthur Erdélyi, who earned his degree of Ph.D. from California Institute of Technology in 1960. Instead of writing on Boehmians and the Boehmian space straightway, we desire to mention, very briefly, some relevant and fruitful thoughts given in the Thesis of Boehme [46]. In his thesis, the finite part of divergent convolution integrals is studied and explored by utilizing Mikusiński's operational calculus (possibly that is the coining of the idea for Boehmians). In Chapters 2 and 3, the concept of an analytic operator function is utilized. An operator function $f(z)$ is said to be an analytic operator function on an open region $\mathbf{S}$ of the complex plane if there is an operator $a \neq 0$ such that $af(z) = \{af(z,t)\}$ has a partial derivative with respect to z, which is continuous on $\mathbf{S} \times 0, \infty)$. Let $f(z)$ be an analytic operator function and $\{f(z,t)\}$ is a continuous function on $S \times [0, \infty)$. Suppose also that for each $t > 0$, $f(z,t)$ is an analytic function on z on larger region $\mathbf{S}^* \supset \mathbf{S}$. Let $f^*(z)$ is an analytic operator function on $\mathbf{S}^*$ such that $f^*(z) = f(z)$ on $\mathbf{S}$. Then the operator function $f^*(z)$ is called [FP $f(z,t)\}$ on $\mathbf{S}^*$.

In fact, the use of the finite parts of divergent integrals started with Cauchy who used, what he called "intégrale extraordinaire", to give a sense to the gamma function for negative values of the argument. This notion has been used and extended by various authors, among them are Schwartz [10] and Lighthill [47] who have applied the theory of distributions to extend the idea of the finite part of divergent integrals. Butzer [48] used the Mikusiński's operational calculus to study the finite part of the divergent convolution integrals.

For certain functions $\{f(z,t)\}$, [cf. Boehme [46, Chap.3]], the finite part of the convolution integral $\int_0^t g(t-u) f(z,u)\,du$ has been defined by Hadamard [49] and Bureau [50] even though for some values of z, the function $\{f(z,t)\}$ is not a Lebesgue integrable function.

The idea of construction of Boehmians is coined from the concept of regular operators introduced by Boehme [51], which form a subalgebra of the field of Mikusiński operators and they, thus, include only such functions whose support is bounded from left. Mikusiński and Mikusiński [52] attempted to generalize the notion of regular operators so as to include all continuous functions and to formulate a general construction of Boehmians. Strictly speaking, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions. Mikusiński [53] introduced and studied the convergence of Boehmians, where the space furnished with the induced convergence, appears to be a complete quasi-normed space. For every ring without zero divisors, there exists the corresponding field of quotients.

The space C^+ of all continuous functions on the real line R with supports bounded from left forms a ring without zero divisors with respect to the convolution. The field of quotients for the space C^+ is called (usually) the field of Mikusiński operators, which is when replaced by the space of all continuous functions C, the construction of the field of quotients becomes impossible due to the presence of zero divisors in C. The construction of Boehmians is similar to that of the field of quotients and in some cases, it is interesting to note, it gives just the field of quotients. On the other hand, the construction of Boehmian is possible where there are zero divisors, such as the space C.

Let G be a linear space and S be the subspace of G. Let to each pair of elements $f \in G$ and $\varphi \in S$, the product $f * g$ is assigned ($*$ is a map from $G \times S$ to G) such that

(i) if $\varphi, \psi \in S$, then $\varphi * \psi \in S$ and $\varphi * \psi = \psi * \varphi$

(ii) if $f \in G, \varphi, \psi \in S$, then $(f * \varphi) * \psi = f * (\varphi * \psi)$

(iii) if $f, g \in G, \varphi \in S$, and $\lambda \in R$, then

$(f + g) * \varphi = f * \varphi + g * \varphi$

and $\lambda(f * \varphi) = (\lambda \ f) * \varphi$.

Let Δ be a family of sequences of elements from S such that

(iv) if $f, g \in G, (\delta_n) \in \Delta$ and $f * \delta_n = g * \delta_n$ $(n = 1,2,\ldots)$, then $f = g$.

(v) if (φ_n), $(\delta_n) \in \Delta$, then $(\varphi_n * \delta_n) \in \Delta$.

Elements of Δ will be called delta sequence. Consider the class A of pair of sequences defined by $\mathsf{A} = \{(f_n), (\varphi_n) : (f_n) \subseteq G^N, (\varphi_n) \in \Delta\}$, for each $n \in N$. An element $((f_n), (\varphi_n)) \in \mathsf{A}$ is called quotient of sequences, denoted by f_n / φ_n, if

$$f_i * \varphi_j = f_j * \varphi_i\ ,\ \forall i, j \in N\ .$$

Two quotients of sequences f_m / φ_m and g_n / ψ_n are called equivalent, denoted by $f_m / \varphi_m \sim g_n / \psi_n$, if

$$f_m * \psi_n = g_n * \varphi_m\ ,\ \forall m, n \in N\ ,$$

which splits A into equivalence classes, of which the class containing f_n / φ_n is denoted by $[f_n / \varphi_n]$. These equivalence classes are called Boehmians and the space of them is denoted by $B = B(G, \Delta)$. Following illustrates the behaviour of Boehmians for the algebraic properties.

(i) The sum of two Boehmians and the multiplication by a scalar are defined by

$$[f_n/\varphi_n]+[g_n/\psi_n]=[(f_n*\psi_n)+(g_n*\varphi_n)/(\varphi_n*\psi_n)]$$

and

$$[\alpha f_n/\varphi_n]=[\alpha f_n/\varphi_n],\ \alpha\in C.$$

(ii) The operation $*$ and the differentiation are, respectively, defined by

$$[f_n/\varphi_n]*g_n/\psi_n]=[(f_n*g_n)/(\varphi_n*\psi_n)]$$

and

$$D^\alpha[f_n/\varphi_n]=[D^\alpha f_n/\varphi_n].$$

In particular, if $[f_n/\varphi_n]\in B$ and $\delta\in S$ is any fixed element, then the product $*$ is defined by

$$[f_n/\varphi_n]*\delta=[(f_n*\delta)/\varphi_n],$$

which is said to be in $B(G,\Delta)$.

More often G, which is also the quasi-normed space, is found to be equipped with the notion of convergence. The intrinsic relationship between this notion of convergence and the product $*$ are given by

(i) if $f_n\to f$ as $n\to\infty$ in G and $\varphi\in S$ be any fixed element, then $f_n*\varphi_n\to f\ *\varphi$ as $n\to\infty$ in G.

(ii) if $f_n\to f$ as $n\to\infty$ in G and $(\delta_n)\in\Delta$, then $f_n*\delta_n\to f$ as $n\to\infty$ in G.

In the Boehmian space B the δ - and Δ - convergences are stated as:

(i) A sequence of Boehmians (x_n) in the Boehmian space B is said to be δ - convergent to a Boehmian x in B, which is denoted by $x_n\xrightarrow{\delta}x$ if there exists a delta sequence (δ_n) such that $(x_n*\delta_n), (x*\delta_n)\in G,\forall n\in N$ and $(x_n*\delta_k)\to(x*\delta_k)$ as $n\to\infty$ in G, $\forall k\in N$.

(ii) A sequence of Boehmians (x_n) in B is said to be Δ- convergent to a Boehmian x in B, denoted by $x_n\xrightarrow{\Delta}x$ if there exists a delta sequence $(\delta_n)\in\Delta$ such that $(x_n-x)*\delta_n\in G,\forall n\in N$ and $(x_n-x)*\delta_n\to 0$ as $n\to\infty$ in G.

Suppose U is an open set. Then a Boehmian $x\in B$ is said to vanish on U if for each compact set $K\subseteq U$ there exists a representative f_n/φ_n of x such that $f_n=0$ on K for each $n\in N$. Thus, the support of a Boehmian x is defined as the complement of the largest open set on which x vanishes. In what follows is an example of a Boehmian space in which the distributions D' can be imbedded.

Consider $G=C^\infty(\mathbb{R})$, which is equipped with the topology of uniform convergence on compact set $S=$ D(R). Let Δ be the class of sequences from D, which satisfies the conditions $\int_{\mathbb{R}}\delta_n(x)xd=1$, $\int_{\mathbb{R}}|\delta_n(x)|\leq M$ and supp $\delta_n\to 0$ as $n\to\infty$. For $f\in G,\varphi\in S$, the convolution $*$ is defined by $(f*\varphi)=\int_{\mathbb{R}}f(x-t)\varphi(t)dt$. Indeed, $*$ defines a map from $G\times S$ to G and a member of Δ satisfies the conditions

(i) if $\alpha,\beta\in G,(\delta_n)\in\Delta$ and $(\alpha*\delta_n)=(\beta*\delta_n)$ for each $n\in N$, then $\alpha=\beta$ in G, and

(ii) if $(\delta_n), (\varphi_n)\in\Delta$, then $(\delta_n*\varphi_n)\in\Delta$,

and thereby generates a Boehmian space, which is $B=B(C^\infty(\mathbb{P}\ ),\Delta)$, members of which are called C^∞ -Boehmians. In another case, consider G to be set of all locally integrable functions on R and identify two such functions, whenever they are equal almost everywhere with respect to the usual Lebesgue measure on $\mathbb{R}$, the topology of which is taken to be the semi-norm topology, generated by

$$p_n(f) = \int_{-n}^{n} |f| d\lambda \ , n = 1,2,\ldots .$$

Also consider $S = \mathrm{D}(\mathrm{R})$ and Δ is the class of sequence from D (discussed in preceding sections). Then a corresponding Boehmian space $B = B(G, \Delta)$ is obtained, called the space of locally (or local) Boehmians. $\mathrm{D}'(\mathrm{R})$ can be imbedded continuously in both the above mentioned Boehmian spaces in the sense that the map $\mathrm{D}' \in B$, given by $u \to (u * \delta_n) / \delta_n]$, defines a one-to-one function in such a way that $u_m \to u$ in D' implies $x_m \to x$ in B, where $x = [(u_m * \delta_n) / \delta_n]$ and $x = [(u * \delta_n) / \delta_n]$.

Mikusiński [54] has constructed a Boehmian space B_{L_1} consisting of integrable Boehmian, on which the Fourier transform is defined as a continuous function. The Boehmian space B_{L_1} is constructed due to $G = \mathrm{L}(\mathrm{R})$ and the class Δ, which satisfies the conditions

$$\int_{\mathbb{R}} \delta_n(x) dx = 1 \ , \ \forall n \in N$$

$\|\delta_n\| < M$, for some $M \in \mathbb{R}$ and all $n \in N$

$\lim_{n\to\infty} \int_{|x|>\varepsilon} |\delta_n(x)| dx = 0$, for each $\varepsilon > 0$

and $(f * \varphi) = \int_{\mathbb{R}^n} f(x-y) g(y) dm(y)$,

where $*$ is the convolution, except for the use of ordinary Lebesgue measure, in place of normalized Lebesgue measure. Mikusiński [54] has also shown that whenever $[f_n/\varphi_n] \in B_{L_1}$,

$$\hat{f}_n(x) = \int_{\mathrm{R}} f_n(t) e^{-itx} dt \ ,$$

converges uniformly on each compact set in R. Then the Fourier transform of an integrable Boehmian $[f_n/\varphi_n]$ is defined as the limit of $\{\hat{f}_n\}$ in the space of continuous functions on R. Mikusiński [55, 56] suggested an extension of space of the Fourier transformable Boehmian containing the tempered distribution S'. The space of tempered Boehmians, which is denoted by B_T, is constructed by taking $G = T$, which is the space of slowly increasing functions on R. Note also that every distributions is the Fourier transform of a tempered Boehmian.

In what follow is the published research work of authors (of this article) related to Boehmians. Looking into both aspects the paucity of space and degree of tolerance of the reader, only the abstract of each paper is given, without destroying the inquisitive thirst of the reader.

1. <u>On the Mellin transform of tempered Boehmians</u>, U.P.B. Sci. Bull. Series A, 62 (4)(2000), 39-48.

Two theorems have been proved on the characteristic theme, that the Mellin transform of tempered Boehmian is a Schwartz distribution. The Mellin transform $\hat{f}(is)$ of slowly increasing function f is the distribution, given by

$$\left\langle \hat{f}(is), \overline{\hat{\varphi}(is)} \right\rangle = 2\pi \left\langle f(e^x), \overline{\varphi(e^x)} \right\rangle .$$

The Mellin transform $\hat{F}$ of tempered Boehmian $F=[f_n/\varphi_n]$ is the limit of $\{\hat{f}_n\}$ in $\mathcal{D}'$ (the space of distributions). Statement of one of the two theorems proved is, *If* $[f_n/\varphi_n]\in B_I$ *, then the sequence* $\{\hat{f}_n\}$ *converges in* $\mathcal{D}'$*. Moreover, if* $[f_n/\varphi_n]=[g_n/\gamma_n]\in B_I$ *, then the sequences* $\{\hat{f}_n\}$ *and* $\{\hat{g}_n\}$ *have the same limit for the Mellin transform of tempered Boehmians*.

2. Wavelet transform of the tempered Boehmians, Hadronic J. Suppl. 18 (2003), 403-410.

This paper deals with the extension of tempered distribution to a class of Boehmians known as tempered Boehmians and defined it on the wavelet transform. Central theme being proving that the wavelet transform of a tempered Boehmian is a distribution, i.e., we have characterized the distributions of the transformable Boehmians. The inversion theorem is also proved. In proving the theorems, continuous Gabor transform (or windowed Fourier transform) of f is used [cf. Debnath [57]] and then the Parseval formula for the Gabor transform is invoked.

3. Wavelet transform for integrable Boehmians (with Lokenath Debnath), J. Math. Anal. Appl. 296 (2) (2004), 473 - 478 .

By applications of continuous wavelet transform and invoking Burzyk's conjecture, the wavelet transform for integrable Boehmians is obtained. Inversion theorem is also proved. Wavelet transform is [cf. Koorwinder [58]]

$$(\Phi_g f)(a,b)=\int f(x)\overline{g_{a,b}(x)}dx=F(a,b) \quad ,$$

where $f\in L^2(\mathbb{R}^d), a\in\mathbb{R}^*, b\in\mathbb{R}^d, \mathbb{R}$ is a set of real numbers, d=1, $\mathbb{R}^*=\mathbb{R}/\{0\}, (\Phi_g f)=(f*h_{a,0})(b)$, and $h(x)=\overline{g(-x)}$.

4. Ultradistribution and ultra-Boehmian of wavelet transform (with S. L. Kalla), Hadronic Journal, 29 (2006), 485-496.

We have investigated certain testing function space for the wavelet transform. Also obtained are ultradistribution and ultra-Boehmians for the wavelet transform. Section 2 deals with the testing function space Z of the wavelet transform, Section 3, based upon the statement (Theorem proved there) that, *the space of all ultradistributions* Z' *contains the space* S' *of tempered distributions*, establishes the result for ultradistrbution of wavelet transform. While extending the wavelet transform to the ultra-Boehmian space in Section 4, it is proved that, *if* $[f_n/\varphi_n]\in\beta_z$*, then the wavelet transform converges in* $\mathcal{D}'$, and further, *if* $[f_n/\varphi_n]=[g_n/\gamma_n]$ *belongs to* β_z*, then the wavelet transform converges to the same limit to which do ultra-Boehmians*.

5. The Cauchy representation of integrable and tempered Boehmians, Kyungpook Math. J. 47 (2007), 481-493.

The paper proves results based on the concept that, a relation between the Cauchy representation of the Fourier transform of the functions in L_2-space and a decomposition of the Fourier transform into two parts, each of which gives an analytic function in the half plane, define that the decomposed transform is convergent for classes of functions larger than those in L_2-space .Section 2 investigates the Cauchy representation of integrable Boehmians by invoking the relation between the Cauchy representation and the Fourier transform and using properties of the former in L_1-space. In Section 3, we have investigated the Cauchy representation of tempered Boehmians. Inversion formulae, for results in Section 2 and 3, are also proved. The conclusive remark of the paper is, the Cauchy representation of an integrable Boehmian and the tempered Boehmian is a distribution.

6. Hilbert Transform for Lacunary Boehmians, Global J. Math. Anal. 1 (1-2)(2007), 85-90.

A series of the form $\sum_{n=-\infty}^{\infty} a_n \exp(i\lambda_n t)$, where $\{\lambda_n\}$ is a sequence of positive integers for which $\inf(\lambda_{n+1}/\lambda_n) > 1$ and $\lambda_{-n} = -\lambda_n$ for all $n \in N$, is called a lacunary series. Nemzer [59] has investigated the space of the lacunary Boehmians, which has lacunary Fourier series representation. In the present paper we study the Hilbert transform for the lacunary Boehmians. A sequence of positive integers $\{\lambda_n\}_{n=1}^{\infty}$ is called Hadamard-lacuanry or simply lacunary if there exists a constant $q > 1$ such that $\lambda_{n+1} > q\lambda_n$ for all n.

7. Mellin transform of fractional integrals for integrable Boehmians, J. Indian Math. Soc. 74 (1-2) (2007), 83-89.

The Riemann-Liouvelle fractional integrals, [Samko et al. [60]], for a function $\varphi(x) \in L_1(a,b)$ are extended from finite interval $[a,b]$ to half axis [Samko et al. [60], page 94] by the formula

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\varphi(t)dt \quad ,0 < x < \infty$$

the Mellin transform of which is [cf. Podlubny [61], page 115]

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{x^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\varphi(x\xi)g(\xi)d\xi \quad .$$

8. Generalized Stieltjes transform and its fractional integrals for integrable Boehmians, Austral. J. Math. Anal. Appl. 5 (1) (2008), 1-8

Using the distributional Stieljes transform and the Parseval relation for the generalized stieltjes transform, in Section 2, a lemma is proved which is further used in proving an important theorem. Section 3 exhibits use of fractional integral operators for integrable Boehmians.Stieltjes transform of fractional integral opeartor is investigated for integrable Boehmians which shows that the Stieltjes transform of fractional integral operator for an integrable Boehmian $F = [f_n/\delta_n]$ is defined as the limit of $(\mathrm{G}_p I_{0+}^{\alpha} f_n)$, which is the space of continuous functions on R.

9. Fourier sine (cosine) transform for ultradistributions and their extensions of tempered and ultra-Boehmian spaces (with S. K. Q. Al-Omari and S. L. Kalla), Integral Transforms Spl. Fuct. 19 (6) (2008), 453-462.

This paper has regarded ultradistributions for the Fourier sine (cosine) transform on certain dual testing space and extension of them on tempered and ultra-Boehmian spaces.

We conclude with a remark that for explanations of notations used and detailed calculations of the results therein, one may refer to original papers, mentioned above. It may not be out of place to mention that some of our work, which are not included above, are viz. Banerji and Loonker [62], Banerji [63], Loonker and Banerji [64, 65, 66, 67] and Banerji and Loonker [68] , Singh et al. [69], Loonker and Banerji [70], Singh and Banerji [71, 72], Loonker and Banerji [73], Singh et al. [74].

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Algunas propiedades de las N-normas (II)

José A. Sarabia R.

En homenaje al Dr. Shyam Kalla

Departamento de Estudios Generales y Básicos, Sección de Matemática
UNEXPO-Barquisimeto
Avenida Corpahuaico, Barquisimeto, 3002, Venezuela
jsarabia196@gmail.com

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Resumen

En el presente trabajo se estudia la posibilidad de solucionar la ecuación funcional de Frank con pares (U,V), donde U es una u-norma y V es una n-norma. También se desarrolla el concepto de a-norma en relación a la ecuación funcional anterior, así como la construcción de operadores TR a partir de n-normas y n-normas k-Lipschitcianas. Finalmente se hacen algunas consideraciones sobre n-normas isomorfas.

Palabras clave: N-norma, a-norma, transformación de relevancia (TR), función k-Lipschitz, n-norma isomorfa.

Some properties of N-norms (II)

Abstract

In this paper we study the possibility of solving the Frank functional equation with pairs (U,V) , where U is an u-norm and V is a n-norm. We also developed the concept of a-norm related with the above functional equation, as well as building up operators RT from n-norms and n-norms k-Lipschitcianas. Finally, some considerations on isomorphic n-norms are made.

Key words: N -norm, a-norm, RT-Transformation, k-Lipschitz functions, isomorphic n-norms.

Introducción

El concepto de n-norma, norma nula o "null norm" es una variante de los conceptos de t-norma, s-norma(t-conorma) y u-norma(uninorma), el cual fue propuesto por T. Calvo en [3]. Su importancia comienza a aumentar, tanto en el campo de las Funciones Asociativas, como en sus aplicaciones, sobre todo en los Sistemas Expertos, Cuantificadores Borrosos, etc. (Vea: [4], [8] y [9]). Inclusive ya se está trabajando en las denominadas n-normas no conmutativas (Vea: [2]). En el presente trabajo se repasan algunos conceptos y ejemplos del trabajo anterior (Vea [13]), con el objeto de establecer la debida conexión entre ellos. Luego se procede a estudiar la posibilidad que existan pares (U,V), donde U es una u – norma y V es una n – norma, tal que solucionen la ecuación de Frank. Como veremos tal cosa no es posible, y por eso cambiamos la n-norma, por una operación binaria más débil, como son las a-normas. Viendo que en este caso, si hay solución a la ecuación funcional. Posteriormente, estudiamos la construcción de operadores

TR, a partir de n-normas, tal como se había hecho en trabajos anteriores, con las t-normas, s-normas y u-normas. Después hablamos de n-normas Lipschitcianas y finalmente, se construyen n-normas isomorfas a una dada.

En este trabajo no incluimos las posibles aproximaciones de Shepard, tal como se hizo en [12], ni estudios de distributividad de n-normas con otras operaciones binarias, y tampoco de las posibles construcciones de implicaciones o co-implicaciones con n-normas, siendo esto, material para un futuro trabajo.

Finalmente, quiero resaltar que este trabajo está especialmente dedicado al **Dr. Shyam Kalla**, eminente matemático, nacido en la India, de magnífica trayectoria en su país, Argentina, y sobre todo en Venezuela, Universidad del Zulia, Kuwait, etc. En todos ellos hizo una gran labor a nivel de post-grado, guiando numerosas tesis de grado a nivel de maestría y doctorado. Y quien, en mi, produjo una gran influencia hacia los diferentes campos de la Matemática Aplicable. ¡Honor a su gran labor!

Preliminares

Daremos algunas definiciones importantes en el desarrollo del tema.

Definición 2.1

Una **t-norma** ***T*** es una operación sobre $I = [0, 1]$, tal que cumple los siguientes axiomas:

(t.1) $T(x,y) = T(y,x)\ \forall x,y \in I$. (Conmutatividad)

(t.2) $T(T(x,y),z) = T\big(x,T(y,z)\big)\ \forall x,y,z \in I$. (Asociatividad)

(t.3) $x \leq y \implies T(x,z) \leq T(y,z)\ \forall x,y,z \in I$. (Monotonía creciente)

(t.4) $T(x,1) = x\ \forall x \in I$. (Existencia del elemento neutro)

Definición 2.2

Una **s-norma** o **t-conorma** ***S*** es una operación sobre $I = [0, 1]$, tal que cumple los siguientes axiomas:

(s.1) $S(x,y) = S(y,x)\ \forall x,y \in I$. (Conmutatividad)

(s.2) $S(S(x,y),z) = S\big(x,S(y,z)\big)\ \forall x,y,z \in I$. (Asociatividad)

(s.3) $x \leq y \implies S(x,z) \leq S(y,z)\ \forall x,y,z \in I$. (Monotonía creciente)

(s.4) $S(x,0) = x\ \forall x \in I$. (Existencia del elemento neutro)

Definición 2.3

Una **u-norma** o **uninorma** *U* es una operación sobre $I = [0, 1]$, tal que cumple los siguientes axiomas:

(u.1) $U(x,y) = U(y,x)\ \forall x,y \in I$. (Conmutatividad)

(u.2) $U(U(x,y),z) = U\big(x,U(y,z)\big)\ \forall x,y,z \in I$. (Asociatividad)

(u.3) $x \leq y \implies U(x,z) \leq U(y,z)\ \forall x,y,z \in I$. (Monotonía creciente)

(u.4) Existe $e \in I$ tal que: $U(x,e) = x\ \forall x \in I$. (Existencia del elemento neutro).

Como vemos, los tres primeros axiomas de cada una, coinciden. Asimismo cuando $e = 0$, tenemos que una u-norma es s-norma, mientras que si $e = 1$, la u-norma es t-norma.

Para un estudio bastante completo de estas operaciones ver: [1], [2], [5], [6], [11] y [12].

Definición 2.4

Una **n-norma** o **norma nula** V es una operación sobre $I = [0, 1]$, tal que cumple los siguientes axiomas:

(n.1) $V(x, y) = V(y, x)\ \forall x, y \in I$. (Conmutatividad)

(n.2) $V(V(x, y), z) = V\big(x, V(y, z)\big)\ \forall x, y, z \in I$. (Asociatividad)

(n.3) $x \leq y \Longrightarrow V(x, z) \leq V(y, z)\ \forall x, y, z \in I$. (Monotonía creciente)

(n.4) Existe $a \in (0,1)$ tal que $V(x, 0) = x\ \forall x \in [0, a]$ y $V(x, 1) = x\ \forall x \in [a, 1]$.

Algunas Propiedades de la n-norma

Para la demostración de las siguientes propiedades de la n-norma, vea [13].

a) $V(x, a) = V(a, x) = a\ \forall x \in I$. O sea a es el **elemento absorbente o aniquilador** de V.

Además, tenemos que: $V(x, y) = a, \forall (x, y) \in ([0, a] \times [a, 1]) \cup ([a, 1] \times [0, a]) = \Re_a$.

b) Si V es una n-norma entonces:

$$T_V(x, y) = \frac{1}{1-a}(V(a + (1-a)x\,, a + (1-a)y) - a)\ (3.1)$$

es una t-norma que recibe el nombre de **t-norma asociada a la n-norma** V.

Asimismo:

$$S_V(x, y) = \frac{1}{a}V(ax, ay)\ (3.2)$$

es una s-norma que recibe el nombre de **s-norma asociada a la n-norma** V.

c) Para una n-norma V de elemento absorbente $a \in (0, 1)$, tenemos que:

$$V(x, y)\begin{cases} \geq \max\{x, y\} \text{ si } (x, y) \in [0, a]^2 \\ \leq \min\{x, y\} \text{ si } (x, y) \in [a, 1]^2 \\ = a \text{ si } (x, y) \in \Re_a \end{cases}$$

Donde: $\Re_a = ([0, a] \times [a, 1]) \cup ([a, 1] \times [0, a])$.

d) Sea V una operación binaria sobre I: conmutativa, asociativa, monótona creciente y cumpliendo la propiedad de valor intermedio en cada componente. Entonces: $V(0, 0) = 0$ y $V(1, 1) = 1$ si y solo si V es t-norma, s-norma, ó n-norma.

e) A partir de una t-norma T y de una s-norma S se puede construir una n-norma V, para cada $a \in I$. En efecto:

Sea T una t-norma, S una s-norma y $a \in (0, 1)$. Entonces existe una única n-norma $V = V_{T,S,a}$ tal que: $T_V = T$, $S_V = \mathrm{T}$ y $V(x, a) = a\ \forall x \in I$. Siendo ésta:

$$V_{T,S,a} = \begin{cases} aS\left(\frac{x}{a}, \frac{y}{a}\right)\ (x, y) \in [0, a]^2 \\ a + (1-a)T\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right)\ (x, y) \in [a, 1]^2\ (3.3) \\ a\ e.o.c \end{cases}$$

Ejemplos de n-normas

Los siguientes ejemplos son construidos usando la propiedad (e). Para mayores detalles y gráficas, vea [13].

1) Sean: $T_P(x,y) = xy$ la t-norma producto y $S_P(x,y) = x + y - xy$ la s-norma producto, y sea $a \in (0, 1)$. Entonces la n-norma definida a partir de ellas, recibe el nombre de **n-norma producto**, y viene dada así:

$$V_{P,a} = \begin{cases} x + y - \dfrac{xy}{a} \; (x,y) \in [0,a]^2 \\ a + \dfrac{(x-a)(y-a)}{1-a} \; (x,y) \in [a,1]^2 \\ a \; (x,y) \in \Re_a \end{cases}$$

2) La siguiente n-norma está construida con un par (T, S), donde ellas no son duales. Así, tomamos: $T_P(x,y) = xy$ (t-norma producto), $S_M(x,y) = \max\{x,y\}$ (s-norma máxima), y $a \in (0, 1)$. Entonces la n-norma construida usando la propiedad (e), es:

$$V_{P,M,a} = \begin{cases} \max\{x,y\} \; (x,y) \in [0,a]^2 \\ a + \dfrac{(x-a)(y-a)}{a} \; (x,y) \in [a,1]^2 \\ a \; (x,y) \in \Re_a \end{cases}$$

Este ejemplo sirve para observar que dada una n-norma V, no siempre T_V y S_V son duales.

3) Sean: $T_L(x,y) = \max\{x + y - 1, 0\}$ (t-norma de Lukasiewicz), $S_L(x,y) = \min\{x + y, 1\}$ (s-norma de Lukasiewicz) y $a \in (0, 1)$. Con ellas definimos la **n-norma de Lukasiewicz,** la cual queda en la forma: (Vea fig.3, con $a = 0.3$ y fig.4, con $a = 0.5$ en [13]).

$$V_{L,a}(x,y) = \begin{cases} x + y \text{ si: } x + y \leq a \wedge (x,y) \in [0,a]^2 \\ x + y - 1 \text{ si: } x + y - 1 \geq a \wedge (x,y) \in [a,1]^2 \\ a \; e.o.c \end{cases}$$

4) Sean: T_D la t-norma drástica y $S_M(x,y) = \max\{x,y\}$, la s-norma máxima. Donde:

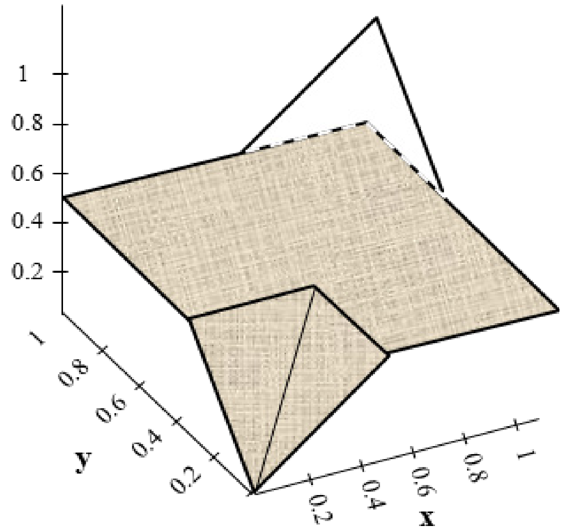
$$T_D(x,y) = \begin{cases} 0 \; (x,y) \in [0,1)^2 \\ \min\{x,y\} \; e.o.c \end{cases}$$

Entonces la n-norma que ellas definen para $a \in (0, 1)$, es:

$$V_{D,M,a}(x,y) = \begin{cases} \max\{x,y\} \text{ si: } (x,y) \in [0,a]^2 \\ \min\{x,y\} \text{ si: } (x = 1 \wedge y \geq a) \vee (y = 1 \wedge x \geq a) \\ a \; e.o.c \end{cases}$$

En la sección 5, veremos que si V es una n-norma cualquiera de elemento absorbente a, entonces: $V_{D,M,a}(x,y) \leq V(x,y) \; \forall (x,y) \in I^2$. Por esta razón a $V_{D,M,a}$ se le llama la **n-norma más débil** de la familia de las n-normas de elemento absorbente a. (Vea la fig. 1, donde $a = 0.5$).

Figura 1



5) Similarmente, si tomamos T_M y S_D. Donde: $T_M(x,y) = \min\{x,y\}$ (t-norma máxima) y

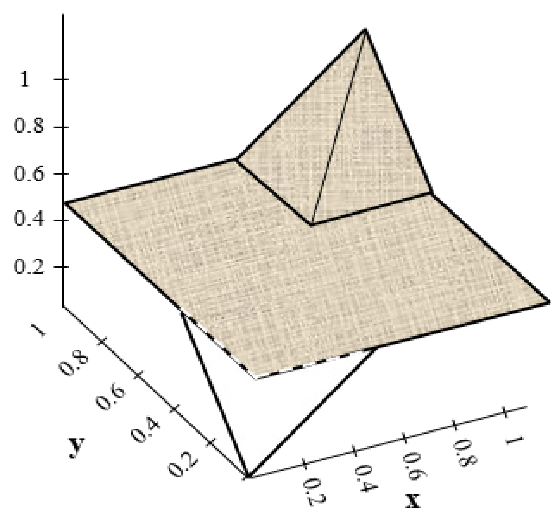
$$S_D(x,y) = \begin{cases} \max\{x,y\} & \text{si: } x.y = 0 \\ 1 & \text{si: } x.y \neq 0 \end{cases}$$ (s-norma drástica).

Entonces, la n-norma que definen: $V_{M,D,a}$, viene dada así

$$V_{M,D,a}(x,y) = \begin{cases} \min\{x,y\} & \text{si: } (x,y) \in [a,1]^2 \\ \max\{x,y\} & \text{si: } \big((x \leq a) \wedge (y = 0)\big) \vee \big((y \leq a) \wedge (x = 0)\big) \\ a & e.o.c \end{cases}$$

En la sección 5, demostraremos que si V es una norma de elemento absorbente a, entonces: $V(x,y) \leq V_{M,D,a}(x,y), \forall (x,y) \in I^2$. Por esta razón a esta n-norma se le conoce con el nombre de la **n-norma más fuerte** (Vea la fig. 2, también con $a = 0.5$)

Figura 2

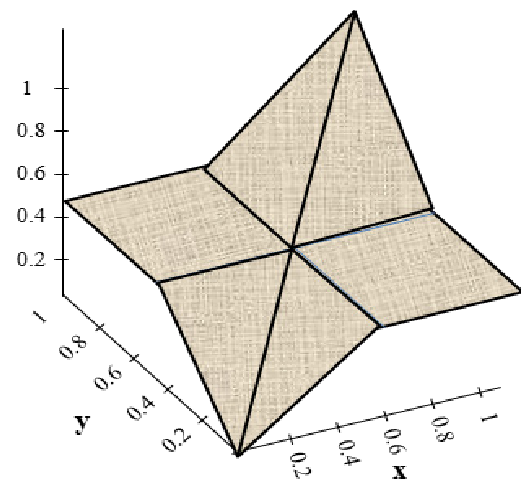


6) Sean ahora: $T_M(x,y) = \min\{x,y\}$, $S_M(x,y) = \max\{x,y\}$ y $a \in (0,1)$. La n-norma que se obtiene de ellos es:

$$V_{min,max,a}(x,y) = \begin{cases} \max\{x,y\} & (x,y) \in [0,a]^2 \\ \min\{x,y\} & (x,y) \in [a,1]^2 \\ a & e.o.c \end{cases}$$

Esta n-norma tiene particular importancia, por ser la única n-norma idempotente con elemento absorbente $a \in (0,1)$ (Vea la fig. 3, donde $a = 0{,}5$).

Figura 3



Otras propiedades de la n-norma

A) Comparación de n-normas de elemento absorbente ***a***

Teorema 5.1

Sea V una n-norma con elemento absorbente $a \in (0,1)$. Entonces:

$$V_{D,M,a}(x,y) \leq V(x,y) \leq V_{M,D,a}(x,y)\ \forall (x,y) \in I^2$$

Demostración:

Vea teorema 3.2 en [13].

B) Dualidad de las n-normas

Definición 5.1

Sea N una función de I en I. Diremos que N es una **negación fuerte** si cumple lo siguiente:

(N.1) $\forall a, b \in I$ con $a < b$, entonces: $N(a) > N(b)$ (Estrictamente decreciente).

(N.2) $\forall a \in I$ se cumple: $N(N(a)) = a$ (Involución).

Algunos ejemplos de negación fuerte son:

1) $N(x) = 1 - x$ (Negación estándar.

2) $N_{S,\beta}(x) = \dfrac{1-x}{1+\beta x}$ $(\beta \neq 0, \beta > -1)$ (Negación de Sugeno de parámetro β)

Entre otras propiedades de la negación fuerte tenemos las siguientes:

a) $N(0) = 1; N(1) = 0$

b) N es biyectiva y continua sobre I.

c) Existe un único punto fijo para N. Por ejemplo para la negación de Sugeno de parámetro β, el punto fijo es: $\bar{a} = \dfrac{\sqrt{1+\beta}-1}{\beta}$.

Teorema 5.2

Sea V una n-norma con elemento absorbente a, y sea N una negación fuerte. Entonces:

$W(x, y) = N\left(V\big(N(x), N(y)\big)\right)$ (5.1), es una n-norma de elemento absorbente $N(a)$.

Demostración:

Vea teorema 3.3 en [13].

Definición 5.2

A la n-norma $W(x, y) = N\left(V\big(N(x), N(y)\big)\right)$ (5.1), se le llama **n-norma dual de** V, y se le denota $W_{V,N}$.

Nota: si $V = W_{V,N}$, es decir si V es auto-dual con respecto a N, entonces: $N(a) = a$. De manera que N debe tener al elemento absorbente a como punto fijo. Luego si: $N(a) \neq a$, entonces V no puede ser auto-dual con respecto a N.

En el teorema siguiente se da un procedimiento para construir una n-norma V que sea auto-dual con respecto a una negación fuerte N. En [7], se menciona un resultado para t-normas que puede ser usado en el mencionado teorema.

Teorema 5.3

Dadas: la t-norma T, la s-norma S y $a \in (0, 1)$, entonces: $V = V_{T,S,a}$ es n- norma auto-dual respecto a la negación fuerte N, si y solo si:

a) $N(a) = a$; b) $S(x, y) = \bar{N}^{-1}\left(T\big(\bar{N}(x), \bar{N}(y)\big)\right)\ \forall x, y \in I$.

Donde: $\bar{N}(t) = \dfrac{N(at) - a}{1-a}\ \forall t \in I$.

Demostración:

Vea teorema 3.4 en [13].

C) Continuidad de las n-normas

La continuidad de una n-norma V está íntimamente ligada a la continuidad de T_V y S_V. En efecto veamos los siguientes resultados (Vea teoremas: 3.5 y 3.6, respectivamente, en [13]).

Teorema 5.4

Sean: T una t-norma continua en I^2, S una s-norma continua en I^2, y $a \in (0,1)$. Entonces: $V_{T,S,a}$ es una n-norma continua en I^2. Recíprocamente, si V es una n-norma continua en I^2, entonces: T_V y S_V, son continuas en I^2.

Teorema 5.5

Sean: T una t-norma y S una s-norma, ambas continuas en I^2. Sea $\langle a_n \rangle$ una sucesión en (0, 1) tal que $a_n \uparrow (\downarrow)\, a \in (0,1)$. Entonces existe

$$\lim_{n\to\infty} V_{T,S,a_n}(x,y) = V(x,y) \;\text{ y } V(x,y) = V_{T,S,a}\,;\; \forall (x,y) \in I^2.$$

Luego V es una n-norma de elemento absorbente a.

Teorema 5.6

Sea $\langle V_n \rangle_{n=1}^{\infty}$ una sucesión de n-normas con elemento absorbente $a_n \in (0,1)$ tal que: $a_n \to a \in (0,1)$ y tal que: $\lim_{n\to\infty} V_n(x,y) = V(x,y)\ \forall (x,y) \in I^2$. Entonces: V es una n-norma de elemento absorbente a.

Demostración:

Por propiedades de límite, es sencillo demostrar que V es una n-norma. Probemos entonces que a es el elemento absorbente de V.

Sea $x \in (0,a)$, luego: $(x,0) \in [0,a)^2$, y como: $a_n \to a$, entonces: existe $m \in N$, tal que: $\forall n \geq m$, se cumple que: $x \leq a_n$. Luego:

$$V(x,0) = \lim_{n\to\infty} V_n(x,0) = \lim_{n\to\infty} a_n S_n\left(\frac{x}{a_n}, 0\right) = x.$$

Asimismo:

$$V(a,0) = \lim_{n\to\infty} V_n(a,0) = \lim_{n\to\infty} a_n S_n\left(\frac{a}{a_n}, 0\right) = a.$$

Luego: $V(x,0) = x\ \forall x \in [0,a]$.

Sea $x \in (a,1]$, entonces existe $m \in N$, tal que: $\forall n \geq m$, se cumple que: $x \geq a_n$, o sea: $(x,1) \in [a_n,1]^2$, y por lo tanto:

$$V(x,1) = \lim_{n\to\infty} V_n(x,1) = \lim_{n\to\infty}\left(a_n + (1-a_n) T_n\left(\frac{x-a_n}{1-a_n}, 1\right)\right) = x.$$

Luego: $V(x,1) = x\ \forall x \in [a,1]$. Por lo tanto: a es el elemento absorbente de V.

D) La ecuación funcional de Frank y las n-normas

Definición 5.3

Diremos que U y V, funciones de I^2 en I, satisfacen la **ecuación funcional de Frank** en I^2, si se cumple que: $U(x,y) + V(x,y) = x + y\ \forall x,y \in I$ (5.2)

En el teorema 5.14 en [5, p.p. 131-132], se demuestra que la t-norma de Frank $T_{F,s}$ y su s-norma dual $S_{F,s}$, satisfacen (4.1). Así, para: $s > 0$ y $s \neq 1$, tenemos:

$$T_{F,s}(x,y) = \mathrm{Log}_s\left(1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right) \quad (5.3)$$

$$S_{F,s}(x,y) = 1 - \mathrm{Log}_s\left(1 + \frac{(s^{1-x} - 1)(s^{1-y} - 1)}{s - 1}\right) \quad (5.4)$$

Siguiendo el resultado anterior, pudiera pensarse que hay algún par (U,V), que satisface (5.2), siendo U una cierta u-norma y V, una n-norma. Sin embargo, veremos que tal par no existe.

Teorema 5.7

Sea U una u-norma con elemento neutro $e \in (0,1)$, y sea V un operador binario creciente en la primera variable, tales que (U,V) satisfacen (5.2) en I^2, entonces V es conmutativo, creciente en cada variable, y e es elemento absorbente de V.

Demostración:

Como (U,V) satisfacen (5.2) en I^2, entonces:

$$V(x,y) = x + y - U(x,y) = y + x - U(y,x) = V(y,x) \ \forall x,y \in I$$

Luego, por la conmutatividad, V es creciente en cada componente.

Además: $V(x,e) = x + e - U(x,e) = x + e - x = e, \forall x \in I.$

Teorema 5.8

a) Sea U una u-norma con elemento neutro $e \in (0, 1)$, entonces no existe una n-norma V tal que el par (U,V) satisfaga (5.2).

b) Sea V una n-norma con elemento absorbente $a \in (0, 1)$, entonces no existe una u-norma U tal que el par (U,V) satisfaga (5.2).

Demostración:

a) Supongamos que existe una n-norma V tal que el par (U,V) satisfaga a (5.2). Por el teorema anterior, e es el elemento absorbente de V, luego: $V(0, e) = e \leq V(0, 1) \leq V(e, 1) = e$, o sea: $V(0, 1) = e$. Pero, entonces: $U(0, 1) = 0 + 1 - V(0, 1) = 1 - e \in (0, 1)$.

Lo que es una contradicción, pues: $U(0, 1) \in \{0, 1\}$, por ser u-norma.

b) Supongamos de nuevo que existe un par (U,V) que satisfaga (5.2), siendo V, una n-norma. Luego: $V(0, 1) = a$, y entonces: $U(0, 1) = 1 - a \in (0, 1)$. Lo que, de nuevo es imposible.

Veremos que si cambiamos la n-norma V, por una operación binaria con menos propiedades, si es posible que haya pares (U,V), donde U es una u-norma, tales que satisfagan (5.2). Por esta razón damos el concepto de a-norma o norma absorbente.

Definición 5.4

Una **a-norma o norma absorbente** A es una función de I^2 en I, tal que:

(a.1) $A(x,y) = A(y,x) \ \forall x,y \in I$. (Conmutativa) (a.2) $A(A(x,y),z) = A(x,A(y,z))$ $\forall x,y,z \in I$. (Asociativa)

(a.3) Existe $a \in I$ tal que: $A(x,a) = a$ $\forall x \in I$. (Existencia de elemento absorbente)

En [10] y [12] se dan algunas propiedades de la a-norma.

Es claro que las t-normas, s-normas y n-normas, son ejemplos de a-normas. Asimismo, si $a \in (0,1)$, una u-norma no puede ser a-norma. Algunos ejemplos de a-normas que no son de los tipos anteriores, son las siguientes: ($a \in (0, 1)$).

1) Definimos la denominada **a-norma más débil.**

$$A_{min}(x,y) = \begin{cases} \max\{x,y\} & (x,y) \in [0,a]^2 \\ 0 & (x,y) \in (a,1]^2 \\ \min\{x,y\} & e.o.c \end{cases}$$

2) La siguiente a-norma es la denominada **a-norma más fuerte**.

$$A_{max}(x,y) = \begin{cases} 1 & (x,y) \in [0,a)^2 \\ \min\{x,y\} & (x,y) \in [a,1]^2 \\ \max\{x,y\} & e.o.c \end{cases}$$

Si A es una a-norma, se cumple que: $A_{min}(x,y) \leq A(x,y) \leq A_{max}(x,y), \forall x,y \in I$.

En el teorema 5 de [10], se hace la afirmación que si U es una u-norma con elemento neutro $e \in (0,1)$, entonces: $A(x,y) = x + y - U(x,y)$, es una a-norma, y por tanto el par (U, A) es una solución de (4.1). Sin embargo esto es **falso**. Veamos el siguiente contraejemplo.

Sea U la u-norma de Hamacher con $\lambda = 1$:

$$U(x,y) = \begin{cases} \dfrac{xy}{xy + (1-x)(1-y)} & \{x,y\} \neq \{0,1\} \\ 0 & e.o.c \end{cases}$$

Entonces $A(x,y)$ viene dada así:

$$A(x,y) = \begin{cases} x + y - \dfrac{xy}{xy + (1-x)(1-y)} & \{x,y\} \neq \{0,1\} \\ 1 & e.o.c \end{cases}$$

Pero A no es asociativa, por tanto no es a-norma. En efecto:

$$A\left(A\left(\frac{1}{3},\frac{1}{4}\right),\frac{2}{3}\right) = \frac{1679}{3388} \neq A\left(\frac{1}{3},A\left(\frac{1}{4},\frac{2}{3}\right)\right) = \frac{893}{1780}$$

Asimismo en [10], se prueba que los pares (U_d, A_{min}), y (U_c, A_{max}) son soluciones de (5.2).

E) Las n-normas y los operadores TR

Los operadores denominados TR o en inglés "RET"(transformaciones de relevancia) son de uso frecuente en la Lógica Borrosa Avanzada, siendo conocida la construcción de estos operadores a partir de t-normas, s-normas y u-normas. En este literal veremos la construcción de operadores TR a partir de n-normas.

Definición 5.5

Un operador binario h sobre I se denomina **transformación de relevancia o TR,** con respecto a $c \in I$, si satisface las siguientes condiciones:

(tr1) $h(0, y) = c$ y $h(1, y) = y \ \forall y \in I$.

(tr2) $\forall y_1, y_2 \in I$ tales que: $y_1 \leq y_2$ se cumple que: $h(x, y_1) \leq h(x, y_2) \ \forall x \in I$.

(tr3) $\forall y \in [c, 1]$ y $\forall x_1, x_2 \in I$ con $x_1 \leq x_2 \implies h(x_1, y) \leq h(x_2, y)$.

(tr4) $\forall y \in [0, c]$ y $\forall x_1, x_2 \in I$ con $x_1 \leq x_2 \implies h(x_1, y) \geq h(x_2, y)$.

Algunas consecuencias de la definición son las siguientes:

1) $h(c, c) = c$.

2) $h(x, c) = c \ \forall x \in I$. Luego, c es elemento absorbente diestro de h.

En lo que sigue daremos ejemplos de operadores TR construidos a partir de t-normas, s-normas y u-normas.

Ejemplos de operadores TR.

1) Sea $T(x, y) = xy$ (t-norma producto) y $c \in I$. Definimos: $h_{T,c}(x, y) = T(x, y) + T(1 - x, c)$. A este operador se le llama **TR producto**.

Este operador TR se puede generalizar de la siguiente manera. Sea σ una función estrictamente creciente sobre I, tal que: $\sigma(0) = 0$ y $\sigma(1) = 1$. El operador: $h(x, y) = \sigma(x)y + \left(1 - \sigma(x)\right)c$, también es un operador TR.

2) Sea S una s-norma, N una negación fuerte y $c = 1$. Entonces: $h_{S,N}(x, y) = S\left(N(x), y\right)$ es un operador TR, denominado operador **TR de implicación**.

3) Sea U la u-norma estándar conjuntiva de elemento neutro $e \in (0, 1)$. O sea:

$$U(x, y) = \begin{cases} \max\{x, y\} & (x, y) \in [e, 1]^2 \\ \min\{x, y\} & e.o.c \end{cases}$$

Entonces el operador: $h_{U,e}(x, y) = U(ex, y) + e(1 - x)$ es un operador TR con respecto a e, llamado **operador TR estándar conjuntivo.**

Operador TR definido a partir de una n-norma

En el teorema siguiente, se da el procedimiento para construir operadores TR a partir de una n-norma y una negación fuerte, y por ende, se está dando un método para obtener muchos ejemplos de operadores TR.

Teorema 5.9

Sea N una negación fuerte con punto fijo $a \in (0,1)$ y V una n-norma con elemento absorbente a. Entonces el operador h definido por:

$$h_V(x, y) = \begin{cases} V(x, y) \ si \ y \in [a, 1] \\ V(N(x), y) \ si \ y \in [0, a] \end{cases} \quad (5.5)$$

Es un operador TR con respecto al elemento a.

Demostración:

(tr1) Sea $y \in [a,1]$, entonces: $h_V(0,y) = V(0,y) = a$, pues: $(0,y) \in \Re_a$.

Similarmente, si: $y \in [0,a]$, entonces: $h_V(0,y) = V(1,y) = a$, pues: $(1,y) \in \Re_a$.

Por otra parte:

$$h_V(1,y) = V(1,y) = a + (1-a)T_V\left(1, \frac{y-a}{1-a}\right) = y \ \forall y \in [a,1]$$

$$h_V(1,y) = V(N(1),y) = V(0,y) = aS_V\left(0, \frac{y}{a}\right) = y \ \forall y \in [0,a]$$

(tr2) Sean: $x, y_1, y_2 \in I$ con $y_1 \leq y_2$. Para probar que h_V satisface este axioma, debemos considerar seis casos:

1) $(x,y_2) \in [0,a]^2$; 2) $(x,y_1) \in [0,a]^2$ y $(x,y_2) \in [0,a] \times [a,1]$;
3) $(x,y_1) \in [0,a] \times [a,1]$; 4) $(x,y_2) \in [a,1] \times [0,a]$;
5) $(x,y_1) \in [a,1] \times [0,a]$ y $(x,y_2) \in [a,1]^2$; 6) $(x,y_1) \in [a,1]^2$.

Demostraremos los casos: 1), 5) y 6). Los demás se prueban en forma parecida.

1) Como $y_1 \leq y_2$ y $(x,y_2) \in [0,a]^2$, entonces: $(x,y_1) \in [0,a]^2$, $(N(x),y_1) \in \Re_a$ y $(N(x),y_2) \in \Re_a$, luego: $h_V(x,y_i) = V(N(x),y_i) = a$ $(i = 1,2)$.

$$5)\ h_V(x,y_1) = V(N(x),y_1) = aS_V\left(\frac{N(x)}{a}, \frac{y_1}{a}\right) \leq a \leq a + (1-a)T_V\left(\frac{x-a}{1-a}, \frac{y_2-a}{1-a}\right) =$$
$$= V(x,y_2) = h_V(x,y_2) \Longrightarrow h_V(x,y_1) \leq h_V(x,y_2)$$

6) Como también: $(x,y_2) \in [a,1]^2$, entonces:

$$h_V(x,y_1) = V(x,y_1) \leq V(x,y_2) = h_V(x,y_2)$$

(tr3) Sean: $y \in [a,1]$ y $\forall x_1, x_2 \in I$ con $x_1 \leq x_2$. Aquí tenemos tres casos:
1) $(x_2,y) \in [0,a] \times [a,1]$; 2) $(x_1,y) \in [0,a] \times [a,1]$ y $(x_2,y) \in [a,1]^2$;
3) $(x_1,y) \in [a,1]^2$. Y la prueba, en cada caso, sigue en forma parecida a lo hecho en (tr2).
(tr4) Sean: $y \in [0,a]$ y $\forall x_1, x_2 \in I$ con $x_1 \leq x_2$. De nuevo tenemos tres casos:
1) $(x_2,y) \in [0,a]^2$; 2) $(x_1,y) \in [0,a]^2$ y $(x_2,y) \in [a,1] \times [0,a]$;
3) $(x_1,y) \in [a,1] \times [0,a]$. Probaremos solamente el caso 3), los otros siguen en forma parecida.

3) En este caso, tenemos que: $(x_2,y) \in [a,1] \times [0,a]$, luego:

$$h_V(x_1,y) = V(N(x_1),y) \geq V(N(x_2),y) = h_V(x_2,y)$$

Por lo tanto h_V es un operador TR.

Ejemplo:

Con este teorema podemos construir una gran cantidad de ejemplos de operadores TR. Por ejemplo, tomando la n-norma producto (Vea ejemplo 1, sección 4), y la negación estándar: $N(x) = 1 - x$. O sea: $a = \frac{1}{2}$. Luego:

$$V(x,y) = \begin{cases} x + y - 2xy\ (x,y) \in [0,1/2]^2 \\ 2xy - x - y + 1\ (x,y) \in [1/2,1]^2 \\ 1/2\ (x,y) \in \Re_{1/2} \end{cases}$$

De esta forma obtenemos:

$$h(x,y) = \begin{cases} 2xy - x - y + 1 & (x,y) \in ([1/2,1] \times [0,1/2]) \cup [1/2,1]^2 \\ 1/2 & e.o.c \end{cases}$$

F) N-normas k-Lipschitcianas

Definición 5.6

Denominamos **función de agregación,** a una función F de I^2 en I, monótona decreciente en cada variable, y tal que: $F(0,0) = 0$ y $F(1,1) = 1$. Por ejemplo, las t-normas, s-normas, u-normas y n-normas, son funciones de agregación.

Asimismo, diremos que una función de agregación F es k - Lipschitz (k - Lips), si existe $k > 0$ tal que:

$$|F(x,y) - F(z,w)| \leq k(|x-z| + |y-w|)\ (5.6)\ \forall (x,y),(z,w) \in I^2$$

Cuando F cumple (4.4) diremos que es ***k* - Lips**. Para un estudio de t-normas k - Lips, vea [2], [3] y [11].

Teorema 5.10

Sea V una n-norma con elemento absorbente $a \in [0,1]$. V es k - Lips si y solo si lo son T_V y S_V.

Demostración:

Supongamos que V es k - Lips. Y sean: $(x, y), (z, w) \in I^2$. Entonces:

$$|T_V(x,y) - T_V(z,w)| =$$

$$\left|\frac{1}{1-a}(V(a+(1-a)x, a+(1-a)y) - a) - \frac{1}{1-a}(V(a+(1-a)z, a+(1-a)w) - a)\right| =$$

$$\frac{1}{1-a}|V(a+(1-a)x, a+(1-a)y) - V(a+(1-a)z, a+(1-a)w)|.$$

Luego: $|T_V(x,y) - T_V(z,w)| \leq k(|x-z| + |y-w|)$

Por lo tanto: T_V es k - Lips. De forma similar, demostramos que S_V es k - Lips.

Recíprocamente, si T_V es k_1 - Lips y S_V es k_2 - Lips, entonces ambos son k - Lips, donde: $k = \max\{k_1, k_2\}$.

Para la demostración tenemos siete casos.

1) $(x,y),(z,w) \in [0,a]^2$

En este caso, tenemos que:

$$|V(x,y)-V(z,w)| = \left|\left(a+(1-a)T\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right)\right)-\left(a+(1-a)T\left(\frac{z-a}{1-a},\frac{w-a}{1-a}\right)\right)\right|$$

$$|V(x,y)-V(z,w)| \leq k(|x-z|+|y-w|)$$

2) $(x,y),(z,w) \in \Re_a$

Simplemente, tenemos: $|V(x,y)-V(z,w)| = |a-a| = 0$

3) $(x,y),(z,w) \in [a,1]^2$

Parecido al caso 1), pero aplicando la definición de S_v.

4) $(x,y) \in [0,a]^2$ y $(z,w) \in [0,a]\times[a,1]$

$$|V(x,y)-V(z,w)| = \left|aS_V\left(\frac{x}{a},\frac{y}{a}\right)-aS_V\left(\frac{z}{a},\frac{a}{a}\right)\right| \leq ak\left(\left|\frac{x}{a}-\frac{z}{a}\right|+\left|\frac{y}{a}-\frac{a}{a}\right|\right) =$$

$$k\big(|x-z|+(a-y)\big) \leq k\big(|x-z|+(w-y)\big) = k(|x-z|+|y-w|)$$

Luego: $|V(x,y)-V(z,w)| \leq k(|x-z|+|y-w|)$

5) $(x,y) \in [0,a]^2$ y $(z,w) \in [a,1]\times[0,a]$

La prueba es similar al caso 4).

6) $(x,y) \in [a,1]\times[0,a]$ y $(z,w) \in [a,1]^2$

$$|V(x,y)-V(z,w)| = \left|a-\left(a+(1-a)T_V\left(\frac{z-a}{1-a},\frac{w-a}{1-a}\right)\right)\right| =$$

$$= (1-a)\left|T_V\left(\frac{x-a}{1-a},\frac{0}{1-a}\right)-T_V\left(\frac{z-a}{1-a},\frac{w-a}{1-a}\right)\right| \leq k(|x-z|+|w-a|) =$$

$$k\big(|x-z|+(w-a)\big) \leq k\big(|x-z|+(w-y)\big) = k(|x-z|+|y-w|)$$

7) $(x,y) \in [0,a]\times[a,1]$ y $(z,w) \in [a,1]^2$

Se prueba en forma similar a 6), usando el hecho que: $T_V(a,0) = 0$.

Luego, V es n-norma k - Lip

N-normas isomorfas

Sea φ una función estrictamente creciente de I sobre I, tal que: $\varphi^{-1}(a) = b$, con $a \in (0,1)$. En lo que sigue veremos cómo se construyen n-normas isomorfas a una n-norma dada V de elemento absorbente a.

Teorema 6.1

Sea φ una función estrictamente creciente de I sobre I, tal que: $\varphi^{-1}(a) = b$, con $a \in (0,1)$, y sea V una n-norma de elemento absorbente a.

Entonces: $V_\varphi(x,y) = \varphi^{-1}\big(V(\varphi(x), \varphi(y))\big)$ es una n-norma de elemento absorbente b.

Demostración:

Como $a \in [0, 1]$, entonces: $b \in (0,1), \varphi([0,b]) = [0,a]$ y $\varphi([b,1]) = [a,1]$.

La conmutatividad de V_φ sigue de la conmutatividad de V. Asimismo, tenemos:

$$V_\varphi\big(V_\varphi(x,y),z\big) = V_\varphi(\varphi^{-1}\big(V(\varphi(x),\varphi(y)),z\big) = \varphi^{-1}\Big(V\Big(V\big(\varphi(x),\varphi(y)\big),\varphi(z)\Big)\Big)$$

$$V_\varphi\big(V_\varphi(x,y),z\big) = \varphi^{-1}\Big(V\Big(\varphi(x),V\big(\varphi(y),\varphi(z)\big)\Big)\Big) = V_\varphi\Big(x,V_\varphi(y,z)\Big).$$

Además, como: φ, φ^{-1} y V son crecientes, se cumple que V_φ es creciente en cada variable.

Por último: $\forall x \in [0, b)$, tenemos que:

$V_\varphi(x, 0) = \varphi^{-1}(V\big(\varphi(x), \varphi(0)\big) = \varphi^{-1}(V(\varphi(x), 0) = \varphi^{-1}\big(\varphi(x)\big) = x$, pues $\varphi([0, b]) = [0, a]$.

Similarmente, como: $\varphi([b,1]) = [a,1]$, entonces: $\forall x \in [b,1]$, se tiene: $V_\varphi(x,1) = x$.

Definición 6.1

A la n-norma V_φ se le llama, **n-norma isomorfa** a V, por medio de φ.

Observaciones:

1) La n-norma V es idempotente, si y solo si, V_φ lo es. Luego, de acuerdo al ejemplo 6, de la sección 4n tenemos:

$$V = V_{min.max,a} \Leftrightarrow V_\varphi = V_{min,max,b}$$

2) Cuando $a = b$, es claro que: $V_\varphi = V = V_{min.max,a}$.

3) Con base en el teorema 6.1, podemos construir numerosos ejemplos de n-normas.

Por ejemplo, sean: $a,b \in (0,1), \varphi(x) = \begin{cases} \dfrac{a}{b^2}x^2 \text{ si } x \in [0,b] \\ \dfrac{a}{b}x \text{ si } x \in [b,1] \end{cases}$ y:

$V(x,y) = \begin{cases} x+y\ (x,y) \in [0,a]^2\ y\ x+y \leq a \\ x+y-1\ (x,y) \in [a,1]^2\ y\ x+y-1 \geq a \\ a\ e.o.c. \end{cases}$. Entonces, tenemos la n-norma:

$$V_\varphi(x,y) = \begin{cases} \sqrt{x^2+y^2}\ (x,y) \in [0,b]^2\ y\ x^2+y^2 \leq b^2 \\ x+y-\dfrac{b}{a}\ (x,y) \in [b,1]^2\ y\ x+y-1 \geq b \\ b\ e.o.c. \end{cases}$$

4) Sean φ y ψ, dos funciones estrictamente crecientes de I sobre I, tales que: $\varphi(b) = a$ y $\psi(c) = b$, donde $a \in [0,1]$, y sea V una n-norma de elemento absorbente a. Entonces se prueba en forma sencilla que: $V_{\varphi \circ \psi} = (V_{\varphi})_{\psi}$ y $(V_{\varphi})_{\varphi^{-1}} = (V_{\varphi^{-1}})_{\varphi} = V$

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Se aceptaran para la publicación artículos científicos inéditos, esto es, que no hayan sido enviados o publicados en otro órgano de divulgación científica con anterioridad, con un máximo de quince (15) páginas y escrito en MICROSOFT OFFICE WORD. Se deben consignar tres copias y un CD-ROM del artículo.

2.1 Artículos de actualización científica que resuman el "Estado del Arte" de las áreas de la revista, con un máximo de quince (15) páginas.

2.2 Artículos de reflexión sobre un tema especifico de las áreas de la revista, desde una perspectiva analítica, interpretativa o crítica del autor, recurriendo a fuentes originales.

2.3 Artículos de revisión: documentos donde se analizan, sistematizan e integran los resultados de investigaciones, publicadas o no, sobre una temática especializada de las áreas de la revista.

2.4 Artículos de discusión: documentos que presentan el debate de afirmaciones realizadas en artículos anteriores.

2.5 Artículos de invitados especiales con un máximo de quince (15) páginas.

2.6 Notas técnicas: sobre mejoras de procesos en ingeniería, resultados de investigaciones derivadas del ejercicio profesional en ingeniería y ciencias afines, con un máximo de ocho (8) páginas.

3. La portada debe contener

3.1 Titulo del trabajo en español e inglés en letras mayúsculas y minúsculas.

3.2 Nombre(s) del autor(s), 6 autores como máximo y su dirección institucional(es) completa(as) (dirección postal, correo electrónico).

3.3 Resumen del trabajo en español e ingles (abstrac) con un máximo de doscientas (200) palabras.

3.4 Palabras clave en español e inglés (key words) con un máximo de cinco (05) palabras.

4. Normas de estilo

4.1 El orden a seguir para el artículo es: portada, resumen (en español e inglés), introducción, fundamentos teóricos, parte experimental, resultados, discusión de resultados, conclusiones, agradecimiento y referencias bibliográficas.

4.2 El artículo debe escribirse en estilo Times New Roman tamaño 12, el título tamaño 16 y las figuras, tablas y texto subordinados tamaño 10.

4.3 El interlineado debe ser doble espacio a excepción del resumen, agradecimiento y referencias bibliográficas que van a un espacio.

4.4 El nombre de cada sección se escribirá en negritas. Estos deberán estar centrados.

4.5 El nombre de las subsecciones se escribirá en negritas a la izquierda del texto y en mayúsculas y minúsculas.

4.6 Las figuras, fotografías, diagramas y gráficos deben denominarse como figuras y éstas deben ir numeradas con números arábigos, así como las tablas. Además, deben incluirse dentro del texto correspondiente y con su respectiva leyenda.

4.7 La redacción de los trabajos puede ser en español o inglés.

4.8 Las fotografías, imágenes, mapas y figuras incluidas en el documento se anexarán también por separado en el CD-ROM con excelente calidad.

4.9 Las fotografías deben estar en blanco y negro, bien contrastadas y brillantes de un ancho máximo de 9,5cm.

4.10 Todos los símbolos matemáticos deben ser escritos en forma clara y legible, con los subíndices y superíndices ubicados correctamente.

4.11 Deben ser numeradas todas las ecuaciones matemáticas en forma consecutiva con números arábigos entre paréntesis y ubicados en el margen derecho.

4.12 Las citas dentro del documento se escriben con el nombre de sus autores seguido con un número entre corchetes que corresponde a la referencia.
Ejemplo: García [1], L Moreno [3, Pág. 19-24] o ubicando el número de la referencia sin colocar autor [4], si son 3 o más autores se escribe K. Nishimoto et al. [2].
Las referencias bibliográficas de artículos de revistas deben contener autor(es) si tiene varios autores separarlos con comas, título de la revista, volumen y número, año de publicación (entre paréntesis) y páginas. Sólo deben incluirse referencias bibliográficas con autores totalmente identificados y se escribirán por orden de aparición de las citas.
Ejemplo: García C., Marín j., Alaña J., Martinez K. y Faria J., Modelo de Nernst-Planck aplicado a la cinética de intercambio iónico del siatema PO4(-3)- Cl(-1) en leche, Rev. Tec. Ing. Univ.Zulia, Vol. 30, Nº. 3, (2007), 236-243.
Los libros se escriben: apellido e inicial del autor(es) con mayúsculas y minúsculas.
Ejemplo: Nishimoto K. (Editor), Fractional Calculus and its Applications, Nihon University, Tokyo (1990).

4.13 En los articulos deben emplearse unidades del Sistema Internacional: metro (m), kilogramos (Kg), Segundo (s), entre otras.

5. Arbitraje

5.1 El trabajo, recibido por el Editor(a), será enviado al Comité Editorial para su revisión y consideración de tres árbitros.

5.2 El método de arbitraje que sigue la revista es de tipo “doble ciego”, que consiste en que tanto los árbitros como los autores desconocen sus respectivas identidades.

5.3 Los trabajos serán evaluados de acuerdo a los siguientes criterios: claridad, originalidad del contenido, aportes al campo de conocimiento, coherencia del discurso, adecuada organización interna, calidad de las referencias bibliográficas, adecuada elaboración del resumen y pertinencia del título.

5.4 En caso de recibir observaciones por parte de los árbitros, estas serán enviadas al(los) autor(es) para su corrección. Después de realizar las correcciones, los autores devolverán la versión corregida en un CD-ROM, así como un original impreso y tres copias.

5.5 Las pruebas de imprentan serán enviadas a los autores para la revisión final del articulo.

5.6 Cuando el trabajo sea aceptado los autores serán informados del tiempo aproximado de publicación. Si el trabajo es rechazado los autores podrán remitir al Editor(a) Jefe una comunicación exponiendo argumentos a favor de su articulo. El Comité Editorial considerará las acciones a tomar.

5.7 Cada autor principal de los artículos publicados recibirá gratuitamente una (1) revista y cinco (5) separatas.

6. Consideraciones finales

6.1 Los conceptos u opiniones emitidos en los artículos, serán de exclusiva responsabilidad de los autores.

6.2 El Comité Editorial se reserva el derecho de los arreglos de estilo que considere conveniente.

6.3 Bajo ninguna circunstancia, la revista devolverá los originales (Impreso y CD-ROM) de los artículos enviados par su consideración.

6.4 Lo no previsto en las presentes normas será resuelto por el Comité Editorial.

Nota: Última revisión aprobada en reunión Nº 70 del Comité Editorial de fecha 11 de Octubre del 2013.