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Integrales que Involucran funciones de Bessel de tres variables y dos parámetros

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Resumen

Las funciones de Bessel tienen aplicaciones en procesos multifotón, en el tratamiento analítico de procesos de campo de iluminación, en el análisis de procesos de dispersión para los cuales la aproximación bipolar no puede ser usada, en el campo de la radiación sincrotón, en el análisis de grandes estructuras, etc. Varias funciones de Bessel generalizadas han sido definidas y estudiadas por diversos autores.En este trabajo se evalúan los siguientes tipos de integrales que involucran funciones de Bessel de tres variables y dos parámetros.

Palabras clave: Funciones de Bessel generalizadas, funciones especiales, integrales.

Integrals involving Bessel functions of three variables and two parameters

Abstract

The Bessel functions have applications in multiphoton processes, in analytical treatment of processes of lighting fields, in analysis of dispersion processes for which the bipolar approximation cannot be used, in the field of synchrotron radiation, in the analysis of big structures, etc. Several generalized Bessel functions have been defined and studied by different authors. In this paper the following types of integrals involving Bessel functions of three variables and two parameters are evaluated.

Key Words: Generalized Bessel functions, special functions, integrals.

Introducción

Las funciones especiales son de suma importancia para científicos e ingenieros debido a sus aplicaciones, en particular las funciones de Bessel aparecen en la solución de ecuaciones diferenciales en matemática, física, química, ingeniería y otras ramas de la ciencia y la tecnología (Galué et al[1]). Las funciones de Bessel tienen aplicaciones en procesos multifotón (Dattoli et al [2]), en el tratamiento analítico de procesos de campo de iluminación especialmente en las teorías de ionización multifotón no-perturbada (Reiss y Krainov [3]), en el análisis de procesos de dispersión para los cuales la aproximación bipolar no puede ser usada (Dattoli et al [4]), en el campo de la radiación sincrotón (Dattoli et al [5]), en el análisis de grandes estructuras [6], etc.

Varias funciones de Bessel generalizadas han sido definidas y estudiadas por diversos autores ([1]-[25]). Entre éstas se tiene la función de Bessel generalizada de tres variables, dos parámetros y un índice (Prieto et al [23]), denotada por $J_n(x,y,z;\tau,\delta)$, la cual puede ser introducida usando la siguiente función generadora:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)+\frac{z}{2}\left(t^3\delta-\frac{1}{t^3\delta}\right)\right]=\sum_{n=\infty}^{+\infty} t^n J_n(x,y,z;\tau,\delta) \tag{1}$$

donde x,y,z son variables reales y t,τ,δ son parámetros complejos $0<|t|,|\tau|,|\delta|<\infty$.

Además, $J_n(x,y,z;\tau,\delta)$ puede ser representada por medio de la serie convergente

$$J_n(x,y,z;\tau,\gamma\tau)=\sum_{m,l=\infty}^{+\infty} \tau^m\gamma^l J_{n-2m-l}(x)J_{m-l}(y)J_l(z) \tag{2}$$

Demostración:

Usando la función generadora dada en (1)

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x,y,z;\tau,\gamma\tau)=\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]\exp\left[\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]\exp\left[\frac{z}{2}\left(t^3\gamma\tau-\frac{1}{t^3\gamma\tau}\right)\right]$$

$$=\sum_{h,k,l=-\infty}^{+\infty} t^{h+2k+3l}\tau^k(\gamma\tau)^l J_h(x)J_k(y)J_l(z)$$

$$=\sum_{h,k,l=-\infty}^{+\infty} t^{(h+l)+2(k+l)}\tau^k(\gamma\tau)^l J_h(x)J_k(y)J_l(z)$$

haciendo cambio de índices

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x,y,z;\tau,\gamma\tau)=\sum_{j,m,l=-\infty}^{+\infty} t^{j+2m}\tau^m\,\gamma^l J_{j-l}(x)J_{m-l}(y)J_l(z)$$

sea $n=j+2m$, entonces

$$\sum_{n=-\infty}^{+\infty} t^n J_n(x,y,z;\tau,\gamma\tau)=\sum_{n,m,l=-\infty}^{+\infty} t^n\tau^m\,\gamma^l J_{n-2m-l}(x)J_{m-l}(y)J_l(z)$$

y al igualar los coeficientes se obtiene el resultado (2).

En este trabajo se evalúan tres tipos de integrales que involucran a la función $J_0(x,y,z;\tau,\delta)$.

A continuación se establece un resultado que será de mucha utilidad en el desarrollo de la próxima sección.

Consideremos el siguiente caso particular de (2):

$$J_0(x,y,z;\tau,\gamma\tau)=\sum_{m,l=-\infty}^{+\infty} \tau^m\,\gamma^l J_{-2m-l}(x)J_{m-l}(y)J_l(z).$$

Separando convenientemente las sumatorias se tiene

$$J_0(x,y,z;\tau,\gamma\tau)=\sum_{m=-\infty}^{-1}\sum_{l=-\infty}^{0} \tau^m\,\gamma^l J_{-2m-l}(x)J_{m-l}(y)J_l(z)+$$

$$\sum_{m=-\infty}^{-1}\sum_{l=1}^{+\infty}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z)+\sum_{m=0}^{+\infty}\sum_{l=-\infty}^{0}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z)+$$

$$\sum_{m=0}^{+\infty}\sum_{l=1}^{+\infty}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z).$$

Haciendo cambios de índices

$$J_0(x,y,z;\tau,\gamma\tau)=\sum_{m=1}^{+\infty}\sum_{l=0}^{+\infty}\tau^{-m}\gamma^{-l}J_{2m+l}(x)J_{-m+l}(y)J_{-l}(z)+$$

$$\sum_{m=1}^{+\infty}\sum_{l=1}^{+\infty}\tau^{-m}\gamma^{l}J_{2m-l}(x)J_{-m-l}(y)J_{l}(z)+\sum_{m=0}^{+\infty}\sum_{l=0}^{+\infty}\tau^{m}\gamma^{-l}J_{-2m+l}(x)J_{m+l}(y)J_{-l}(z)+$$

$$\sum_{m=0}^{+\infty}\sum_{l=1}^{+\infty}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z),$$

el cual puede escribirse como

$$J_0(x,y,z;\tau,\gamma\tau)=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}\tau^{-m}\gamma^{-l}J_{2m+l}(x)J_{-m+l}(y)J_{-l}(z)+$$

$$\sum_{m=1}^{+\infty}\sum_{l=m+1}^{+\infty}\tau^{-m}\gamma^{-l}J_{2m+l}(x)J_{-m+l}(y)J_{-l}(z)+\sum_{m=1}^{+\infty}\sum_{l=1}^{2m}\tau^{-m}\gamma^{l}J_{2m-l}(x)J_{-m-l}(y)J_{l}(z)+$$

$$\sum_{m=1}^{+\infty}\sum_{l=2m+1}^{+\infty}\tau^{-m}\gamma^{l}J_{2m-l}(x)J_{-m-l}(y)J_{l}(z)+\sum_{m=0}^{+\infty}\sum_{l=0}^{2m}\tau^{m}\gamma^{-l}J_{-2m+l}(x)J_{m+l}(y)J_{-l}(z)+$$

$$\sum_{m=0}^{+\infty}\sum_{l=2m+1}^{+\infty}\tau^{m}\gamma^{-l}J_{-2m+l}(x)J_{m+l}(y)J_{-l}(z)+\sum_{l=1}^{+\infty}\sum_{m=0}^{l}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z)+$$

$$\sum_{l=1}^{+\infty}\sum_{m=l+1}^{+\infty}\tau^{m}\gamma^{l}J_{-2m-l}(x)J_{m-l}(y)J_{l}(z),$$

donde se intercambió el orden de sumación de la última serie doble sobre la base de la convergencia absoluta. Ahora, usando la bien conocida propiedad de simetría de las funciones de Bessel (Lebedev, Special [26])

$J_{-n}(x)=(-1)^{n}J_{n}(x),\qquad n=1,2,3,\cdots$ se obtiene

$$J_0(x,y,z;\tau,\gamma\tau)=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^{m}\tau^{-m}\gamma^{-l}J_{2m+l}(x)J_{m-l}(y)J_{l}(z)+$$

$$\sum_{m=1}^{+\infty}\sum_{l=m+1}^{+\infty}(-1)^{l}\tau^{-m}\gamma^{-l}J_{2m+l}(x)J_{-m+l}(y)J_{l}(z)+\sum_{m=1}^{+\infty}\sum_{l=1}^{2m}(-1)^{m+l}\tau^{-m}\gamma^{l}J_{2m-l}(x)J_{m+l}(y)J_{l}(z)+$$

$$\sum_{m=1}^{+\infty}\sum_{l=2m+1}^{+\infty}(-1)^{-m}\tau^{-m}\gamma^{l}J_{-2m+l}(x)J_{m+l}(y)J_{l}(z)+\sum_{m=0}^{+\infty}\sum_{l=0}^{2m}\tau^{m}\gamma^{-l}J_{2m-l}(x)J_{m+l}(y)J_{l}(z)+$$

$$\sum_{m=0}^{+\infty}\sum_{l=2m+1}^{+\infty}(-1)^{l}\tau^{m}\gamma^{-l}J_{-2m+l}(x)J_{m+l}(y)J_{l}(z)+\sum_{l=1}^{+\infty}\sum_{m=0}^{l}(-1)^{m}\tau^{m}\gamma^{l}J_{2m+l}(x)J_{-m+l}(y)J_{l}(z)+$$

$$\sum_{l=1}^{+\infty}\sum_{m=l+1}^{+\infty}(-1)^{l}\tau^{m}\gamma^{l}J_{2m+l}(x)J_{m-l}(y)J_{l}(z), \qquad (3)$$

Cálculo de Integrales

Integrales de la forma $\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz}J_0\left(\sqrt{x},\sqrt{y},\sqrt{z};\tau,\gamma\tau\right)dx\,dy\,dz,\ \mathrm{Re}(a)>\mathbf{0},$

$\mathrm{Re}(b)>\mathbf{0},\mathrm{Re}(c)>\mathbf{0}$:

Sustituyendo $J_0\left(\sqrt{x},\sqrt{y},\sqrt{z};\tau,\gamma\tau\right)$ por su representación en serie dada en (3)

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz}J_0\left(\sqrt{x},\sqrt{y},\sqrt{z};\tau,\gamma\tau\right)dx\,dy\,dz=$$

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz}\left\{\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^m\,\tau^{-m}\,\gamma^{-l}J_{2m+l}(\sqrt{x})J_{m-l}(\sqrt{y})J_l(\sqrt{z})+\right.$$

$$\sum_{m=1}^{+\infty}\sum_{l=m+1}^{+\infty}(-1)^l\,\tau^{-m}\,\gamma^{-l}J_{2m+l}(\sqrt{x})J_{-m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\sum_{m=1}^{+\infty}\sum_{l=1}^{2m}(-1)^{m+l}\,\tau^{-m}\,\gamma^{l}J_{2m-l}(\sqrt{x})J_{m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\sum_{m=1}^{+\infty}\sum_{l=2m+1}^{+\infty}(-1)^{-m}\,\tau^{-m}\,\gamma^{l}J_{-2m+l}(\sqrt{x})J_{m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\sum_{m=0}^{+\infty}\sum_{l=0}^{2m}\tau^{m}\,\gamma^{-l}J_{2m-l}(\sqrt{x})J_{m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\sum_{m=0}^{+\infty}\sum_{l=2m+1}^{+\infty}(-1)^l\,\tau^{m}\,\gamma^{-l}J_{-2m+l}(\sqrt{x})J_{m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\sum_{l=1}^{+\infty}\sum_{m=0}^{l}(-1)^m\,\tau^{m}\,\gamma^{l}J_{2m+l}(\sqrt{x})J_{-m+l}(\sqrt{y})J_l(\sqrt{z})+$$

$$\left.\sum_{l=1}^{+\infty}\sum_{m=l+1}^{+\infty}(-1)^l\,\tau^{m}\,\gamma^{l}J_{2m+l}(\sqrt{x})J_{m-l}(\sqrt{y})J_l(\sqrt{z})\right\}dx\,dy\,dz. \tag{4}$$

Sea

$$I_1=\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz}\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^m\,\tau^{-m}\,\gamma^{-l}J_{2m+l}(\sqrt{x})J_{m-l}(\sqrt{y})J_l(\sqrt{z})\,dx\,dy\,dz.$$

Ahora, intercambiando el orden de las integrales y las sumatorias, asumiendo que las integrales son absolutamente convergentes, se obtiene

$$I_1=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^m\,\tau^{-m}\,\gamma^{-l}\left(\int_0^\infty e^{-ax}J_{2m+l}(\sqrt{x})\,dx\right)\left(\int_0^\infty e^{-by}J_{m-l}(\sqrt{y})\,dy\right)\times$$
$$\left(\int_0^\infty e^{-cz}J_l(\sqrt{z})\,dz\right).$$

La sustitución de cada función de Bessel por su representación en serie (Lebedev, N., Special [26, Pág. 102, No. (5.3.2)])

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)},\qquad |z|<\infty,\ |\arg z|<\pi,\ \nu\in\mathrm{C} \tag{5}$$

conduce a la expresión

$$I_1=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^m\,\tau^{-m}\,\gamma^{-l}\left(\sum_{k_1=0}^{+\infty}\frac{(-1)^{k_1}}{2^{2m+l+2k_1}\,k_1!\,\Gamma(2m+l+k_1+1)}\int_0^\infty e^{-ax}\,x^{\frac{2m+l}{2}+k_1}\,dx\right)\times$$

$$\left(\sum_{k_2=0}^{+\infty}\frac{(-1)^{k_2}}{2^{m-l+2k_2}\,k_2!\,\Gamma(m-l+k_2+1)}\int_0^\infty e^{-by}\,y^{\frac{m-l}{2}+k_2}\,dy\right)\times$$

$$\left(\sum_{k_3=0}^{+\infty}\frac{(-1)^{k_3}}{2^{l+2k_3}\,k_3!\,\Gamma(l+k_3+1)}\int_0^\infty e^{-cz}\,z^{\frac{l}{2}+k_3}\,dz\right).$$

Evaluando las integrales mediante el siguiente resultado [26, p. 13, No. (1.5.1)]

$$\int_0^\infty e^{-pt}t^{z-1}dt=\frac{\Gamma(z)}{p^z},\ \mathrm{Re}(p)>0,\ \mathrm{Re}(z)>0, \tag{6}$$

se obtiene

$$I_1=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}(-1)^m\,\tau^{-m}\,\gamma^{-l}\left(\sum_{k_1=0}^{+\infty}\frac{(-1)^{k_1}\,\Gamma\left(\frac{2m+l}{2}+k_1+1\right)}{2^{2m+l+2k_1}\,a^{\frac{2m+l}{2}+k_1+1}k_1!\,\Gamma(2m+l+k_1+1)}\right)\times$$

$$\left(\sum_{k_2=0}^{+\infty}\frac{(-1)^{k_2}\,\Gamma\left(\frac{m-l}{2}+k_2+1\right)}{2^{m-l+2k_2}\,b^{\frac{m-l}{2}+k_2+1}\,k_2!\,\Gamma(m-l+k_2+1)}\right)\left(\sum_{k_3=0}^{+\infty}\frac{(-1)^{k_3}\,\Gamma\left(\frac{l}{2}+k_3+1\right)}{2^{l+2k_3}\,c^{\frac{l}{2}+k_3+1}\,k_3!\,\Gamma(l+k_3+1)}\right).$$

Al aplicar la definición de la función hipergeométrica ${}_pF_q(\cdot)$ [26, p. 275, No. (9.14.2)] la expresión anterior puede escribirse en la forma siguiente

$$I_1=\sum_{m=1}^{+\infty}\sum_{l=0}^{m}\frac{(-1)^m\,\Gamma\left(\frac{2m+l}{2}+1\right)\Gamma\left(\frac{m-l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\,\tau^{-m}\,\gamma^{-l}}{2^{3m+l}\,\Gamma(2m+l+1)\Gamma(m-l+1)\Gamma(l+1)\,a^{\frac{2m+l}{2}+1}\,b^{\frac{m-l}{2}+1}\,c^{\frac{l}{2}+1}}\times$$

$${}_1F_1\left(\frac{2m+l}{2}+1;2m+l+1;-\frac{1}{4a}\right)\,{}_1F_1\left(\frac{m-l}{2}+1;m-l+1;-\frac{1}{4b}\right)\times$$

$${}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right). \tag{7}$$

Similarmente se evalúan las otras integrales y finalmente se obtiene

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-ax-by-cz}J_0\left(\sqrt{x},\sqrt{y},\sqrt{z};\tau,\gamma\tau\right)dx\,dy\,dz=$$

$$\sum_{m=1}^{+\infty}\sum_{l=0}^{m}\left[\frac{(-1)^{m}\,\Gamma\left(\frac{2m+l}{2}+1\right)\Gamma\left(\frac{m-l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\,\tau^{-m}\,\gamma^{-l}}{2^{3m+l}\,\Gamma(2m+l+1)\Gamma(m-l+1)\Gamma(l+1)\,a^{\frac{2m+l}{2}+1}\,b^{\frac{m-l}{2}+1}\,c^{\frac{l}{2}+1}}\right.\times$$

$${}_1F_1\left(\frac{2m+l}{2}+1;2m+l+1;-\frac{1}{4a}\right)\,{}_1F_1\left(\frac{m-l}{2}+1;m-l+1;-\frac{1}{4b}\right)\times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=m+1}^{+\infty}\left[\frac{(-1)^{l}\,\Gamma\left(\frac{2m+l}{2}+1\right)\Gamma\left(\frac{-m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\,\tau^{-m}\,\gamma^{-l}}{2^{m+3l}\,\Gamma(2m+l+1)\Gamma(-m+l+1)\Gamma(l+1)\,a^{\frac{2m+l}{2}+1}\,b^{\frac{-m+l}{2}+1}\,c^{\frac{l}{2}+1}}\right.\times$$

$${}_1F_1\left(\frac{2m+l}{2}+1;2m+l+1;-\frac{1}{4a}\right)\,{}_1F_1\left(\frac{-m+l}{2}+1;-m+l+1;-\frac{1}{4b}\right)\times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=1}^{2m}\left[\frac{(-1)^{m+l}\,\Gamma\left(\frac{2m-l}{2}+1\right)\Gamma\left(\frac{m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\,\tau^{-m}\,\gamma^{l}}{2^{3m+l}\,\Gamma(2m-l+1)\Gamma(m+l+1)\Gamma(l+1)\,a^{\frac{2m-l}{2}+1}\,b^{\frac{m+l}{2}+1}\,c^{\frac{l}{2}+1}}\right.\times$$

$${}_1F_1\left(\frac{2m-l}{2}+1;2m-l+1;-\frac{1}{4a}\right)\,{}_1F_1\left(\frac{m+l}{2}+1;m+l+1;-\frac{1}{4b}\right)\times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=2m+1}^{+\infty}\left[\frac{(-1)^{-m}\,\Gamma\left(\frac{-2m+l}{2}+1\right)\Gamma\left(\frac{m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\,\tau^{-m}\,\gamma^{l}}{2^{-m+3l}\,\Gamma(-2m+l+1)\Gamma(m+l+1)\Gamma(l+1)\,a^{\frac{-2m+l}{2}+1}\,b^{\frac{m+l}{2}+1}\,c^{\frac{l}{2}+1}}\right.\times$$

$${}_1F_1\left(\frac{-2m+l}{2}+1;-2m+l+1;-\frac{1}{4a}\right)\,{}_1F_1\left(\frac{m+l}{2}+1;m+l+1;-\frac{1}{4b}\right)\times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{m=0}^{+\infty}\sum_{l=0}^{2m}\left[\frac{\Gamma\left(\frac{2m-l}{2}+1\right)\Gamma\left(\frac{m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\ \tau^{m}\ \gamma^{-l}}{2^{3m+l}\ \Gamma(2m-l+1)\Gamma(m+l+1)\Gamma(l+1)\ a^{\frac{2m-l}{2}+1}\ b^{\frac{m+l}{2}+1}\ c^{\frac{l}{2}+1}}\right. \times$$

$${}_1F_1\left(\frac{2m-l}{2}+1;2m-l+1;-\frac{1}{4a}\right)\ {}_1F_1\left(\frac{m+l}{2}+1;m+l+1;-\frac{1}{4b}\right) \times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{m=0}^{+\infty}\sum_{l=2m+1}^{+\infty}\left[\frac{(-1)^{l}\ \Gamma\left(\frac{-2m+l}{2}+1\right)\Gamma\left(\frac{m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\ \tau^{m}\ \gamma^{-l}}{2^{-m+3l}\ \Gamma(-2m+l+1)\Gamma(m+l+1)\Gamma(l+1)\ a^{\frac{-2m+l}{2}+1}\ b^{\frac{m+l}{2}+1}\ c^{\frac{l}{2}+1}}\right. \times$$

$${}_1F_1\left(\frac{-2m+l}{2}+1;-2m+l+1;-\frac{1}{4a}\right)\ {}_1F_1\left(\frac{m+l}{2}+1;m+l+1;-\frac{1}{4b}\right) \times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{l=1}^{+\infty}\sum_{m=0}^{l}\left[\frac{(-1)^{m}\ \Gamma\left(\frac{2m+l}{2}+1\right)\Gamma\left(\frac{-m+l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\tau^{m}\ \gamma^{l}}{2^{m+3l}\ \Gamma(2m+l+1)\Gamma(-m+l+1)\Gamma(l+1)\ a^{\frac{2m+l}{2}+1}\ b^{\frac{-m+l}{2}+1}\ c^{\frac{l}{2}+1}}\right. \times$$

$${}_1F_1\left(\frac{2m+l}{2}+1;2m+l+1;-\frac{1}{4a}\right)\ {}_1F_1\left(\frac{-m+l}{2}+1;-m+l+1;-\frac{1}{4b}\right) \times$$

$$\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]+$$

$$\sum_{l=1}^{+\infty}\sum_{m=l+1}^{+\infty}\left[\frac{(-1)^{l}\ \Gamma\left(\frac{2m+l}{2}+1\right)\Gamma\left(\frac{m-l}{2}+1\right)\Gamma\left(\frac{l}{2}+1\right)\tau^{m}\ \gamma^{l}}{2^{3m+l}\ \Gamma(2m+l+1)\Gamma(m-l+1)\Gamma(l+1)\ a^{\frac{2m+l}{2}+1}\ b^{\frac{m-l}{2}+1}\ c^{\frac{l}{2}+1}}\right. \times$$

$${}_1F_1\left(\frac{2m+l}{2}+1;2m+l+1;-\frac{1}{4a}\right)\ {}_1F_1\left(\frac{m-l}{2}+1;m-l+1;-\frac{1}{4b}\right) \times$$

$$\left.\left.{}_1F_1\left(\frac{l}{2}+1;l+1;-\frac{1}{4c}\right)\right]\right\}. \qquad (8)$$

Integrales de la forma $\int_0^1\int_0^1\int_0^1\left(1-x^2\right)^{\mu-1}\left(1-y^2\right)^{\nu-1}\left(1-z^2\right)^{\omega-1} \times$

$J_0\left(\sqrt{1-x^2},\sqrt{1-y^2},\sqrt{1-z^2};\tau,\gamma\tau\right)dx\,dy\,dz$, $\mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\omega)>0$:

Usando un procedimiento similar al aplicado en la sección anterior y empleando en lugar de (6) el resultado (Gradshteyn, I. S. and Ryzhik [27, p. 294, No. (3.249.5)])

$$\int_0^1\left(1-x^2\right)^{\mu-1}dx=\frac{1}{2}B\left(\frac{1}{2},\mu\right)=\frac{\sqrt{\pi}}{2}\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})},\ \mathrm{Re}(\mu)>0$$

se tiene:

$$\int_0^1\int_0^1\int_0^1\left(1-x^2\right)^{\mu-1}\left(1-y^2\right)^{\nu-1}\left(1-z^2\right)^{\omega-1}J_0\left(\sqrt{1-x^2},\sqrt{1-y^2},\sqrt{1-z^2};\tau,\gamma\tau\right)dx\,dy\,dz$$

$$=\left(\frac{\sqrt{\pi}}{2}\right)^3\left\{\sum_{m=1}^{\infty}\sum_{l=0}^{m}\left[\frac{(-1)^m\,\Gamma\left(\mu+\frac{2m+l}{2}\right)\Gamma\left(\nu+\frac{m-l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^{-m}\gamma^{-l}}{2^{3m+l}\,\Gamma\left(\mu+\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\nu+\frac{m-l}{2}+\frac{1}{2}\right)\Gamma(m-l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\right.\times$$

$${}_1F_2\left(\mu+\tfrac{2m+l}{2};\mu+\tfrac{2m+l}{2}+\tfrac{1}{2},2m+l+1;-\tfrac{1}{4}\right)\times$$

$${}_1F_2\left(\nu+\tfrac{m-l}{2};\nu+\tfrac{m-l}{2}+\tfrac{1}{2},m-l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+$$

$$\sum_{m=1}^{\infty}\sum_{l=m+1}^{\infty}\left[\frac{(-1)^l\,\Gamma\left(\mu+\frac{2m+l}{2}\right)\Gamma\left(\nu+\frac{l-m}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^{-m}\gamma^{-l}}{2^{m+3l}\,\Gamma\left(\mu+\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\nu+\frac{l-m}{2}+\frac{1}{2}\right)\Gamma(-m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$${}_1F_2\left(\mu+\tfrac{2m+l}{2};\mu+\tfrac{2m+l}{2}+\tfrac{1}{2},2m+l+1;-\tfrac{1}{4}\right)\times$$

$${}_1F_2\left(\nu+\tfrac{l-m}{2};\nu+\tfrac{l-m}{2}+\tfrac{1}{2},-m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+$$

$$\sum_{m=1}^{\infty}\sum_{l=1}^{2m}\left[\frac{(-1)^{m+l}\,\Gamma\left(\mu+\frac{2m-l}{2}\right)\Gamma\left(\nu+\frac{m+l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^{-m}\gamma^{l}}{2^{3m+l}\,\Gamma\left(\mu+\frac{2m-l}{2}+\frac{1}{2}\right)\Gamma(2m-l+1)\Gamma\left(\nu+\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$${}_1F_2\left(\mu+\tfrac{2m-l}{2};\mu+\tfrac{2m-l}{2}+\tfrac{1}{2},2m-l+1;-\tfrac{1}{4}\right)\times$$

$${}_1F_2\left(\nu+\tfrac{m+l}{2};\nu+\tfrac{m+l}{2}+\tfrac{1}{2},m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+$$

$$\sum_{m=1}^{\infty}\sum_{l=2m+1}^{\infty}\left[\frac{(-1)^{-m}\,\Gamma\left(\mu+\frac{l-2m}{2}\right)\Gamma\left(\nu+\frac{m+l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^{-m}\gamma^{l}}{2^{-m+3l}\,\Gamma\left(\mu+\frac{l-2m}{2}+\frac{1}{2}\right)\Gamma(-2m+l+1)\Gamma\left(\nu+\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$$ {}_1F_2\left(\mu+\tfrac{l-2m}{2};\mu+\tfrac{l-2m}{2}+\tfrac{1}{2},-2m+l+1;-\tfrac{1}{4}\right)\times $$

$$ {}_1F_2\left(\nu+\tfrac{m+l}{2};\nu+\tfrac{m+l}{2}+\tfrac{1}{2},m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+ $$

$$ \sum_{m=0}^{\infty}\sum_{l=0}^{2m}\left[\frac{\Gamma\left(\mu+\frac{2m-l}{2}\right)\Gamma\left(\nu+\frac{m+l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^m\gamma^{-l}}{2^{3m+l}\,\Gamma\left(\mu+\frac{2m-l}{2}+\frac{1}{2}\right)\Gamma(2m-l+1)\Gamma\left(\nu+\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times $$

$$ {}_1F_2\left(\mu+\tfrac{2m-l}{2};\mu+\tfrac{2m-l}{2}+\tfrac{1}{2},2m-l+1;-\tfrac{1}{4}\right)\times $$

$$ {}_1F_2\left(\nu+\tfrac{m+l}{2};\nu+\tfrac{m+l}{2}+\tfrac{1}{2},m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+ $$

$$ \sum_{m=0}^{\infty}\sum_{l=2m+1}^{\infty}\left[\frac{(-1)^l\,\Gamma\left(\mu+\frac{l-2m}{2}\right)\Gamma\left(\nu+\frac{m+l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^m\gamma^{-l}}{2^{-m+3l}\,\Gamma\left(\mu+\frac{l-2m}{2}+\frac{1}{2}\right)\Gamma(-2m+l+1)\Gamma\left(\nu+\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times $$

$$ {}_1F_2\left(\mu+\tfrac{l-2m}{2};\mu+\tfrac{l-2m}{2}+\tfrac{1}{2},-2m+l+1;-\tfrac{1}{4}\right)\times $$

$$ {}_1F_2\left(\nu+\tfrac{m+l}{2};\nu+\tfrac{m+l}{2}+\tfrac{1}{2},m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+ $$

$$ \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left[\frac{(-1)^m\,\Gamma\left(\mu+\frac{2m+l}{2}\right)\Gamma\left(\nu+\frac{l-m}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^m\gamma^{l}}{2^{m+3l}\,\Gamma\left(\mu+\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\nu+\frac{l-m}{2}+\frac{1}{2}\right)\Gamma(-m+l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times $$

$$ {}_1F_2\left(\mu+\tfrac{2m+l}{2};\mu+\tfrac{2m+l}{2}+\tfrac{1}{2},2m+l+1;-\tfrac{1}{4}\right)\times $$

$$ {}_1F_2\left(\nu+\tfrac{l-m}{2};\nu+\tfrac{l-m}{2}+\tfrac{1}{2},-m+l+1;-\tfrac{1}{4}\right){}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]+ $$

$$ \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty}\left[\frac{(-1)^l\,\Gamma\left(\mu+\frac{2m+l}{2}\right)\Gamma\left(\nu+\frac{m-l}{2}\right)\Gamma\left(\omega+\frac{l}{2}\right)\tau^m\gamma^{l}}{2^{3m+l}\,\Gamma\left(\mu+\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\nu+\frac{m-l}{2}+\frac{1}{2}\right)\Gamma(m-l+1)\Gamma\left(\omega+\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times $$

$$ {}_1F_2\left(\mu+\tfrac{2m+l}{2};\mu+\tfrac{2m+l}{2}+\tfrac{1}{2},2m+l+1;-\tfrac{1}{4}\right)\times $$

$$ {}_1F_2\left(\nu+\tfrac{m-l}{2};\nu+\tfrac{m-l}{2}+\tfrac{1}{2},m-l+1;-\tfrac{1}{4}\right)\;{}_1F_2\left(\omega+\tfrac{l}{2};\omega+\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{1}{4}\right)\Big]\Big\}. \tag{9} $$

Integralesde la forma $\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\sqrt{(x+a)(x+b)}}\frac{1}{\sqrt{(y+a)(y+b)}}\frac{1}{\sqrt{(z+a)(z+b)}}\times$

$J_0\left(\frac{\sqrt{x}}{\sqrt{(x+a)(x+b)}},\frac{\sqrt{y}}{\sqrt{(y+a)(y+b)}},\frac{\sqrt{z}}{\sqrt{(z+a)(z+b)}};\tau,\gamma\tau\right)dx\,dy\,dz,\ a,b\in R^+:$

Análogamente a las secciones anteriores y empleando en lugar de (6) el resultado [27, p. 286, No. (3.197.7)]

$$\int_0^\infty x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu}dx=\sqrt{\pi}\left(\sqrt{a}+\sqrt{b}\right)^{1-2\mu}\frac{\Gamma\left(\mu-\frac{1}{2}\right)}{\Gamma(\mu)},\ \mathrm{Re}(\mu)>0$$

se tiene:

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{\sqrt{(x+a)(x+b)}}\frac{1}{\sqrt{(y+a)(y+b)}}\frac{1}{\sqrt{(z+a)(z+b)}}\,J_0\left(\frac{\sqrt{x}}{\sqrt{(x+a)(x+b)}},\frac{\sqrt{y}}{\sqrt{(y+a)(y+b)}},\frac{\sqrt{z}}{\sqrt{(z+a)(z+b)}};\tau,\gamma\tau\right)dx\,dy\,($$

$$\left(\sqrt{\pi}\right)^3\left\{\sum_{m=1}^{+\infty}\sum_{l=0}^{m}\left[\frac{(-1)^m\,\Gamma\left(\frac{2m+l}{2}\right)\Gamma\left(\frac{m-l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-3m-l}\tau^{-m}\gamma^{-l}}{2^{3m+l}\,\Gamma\left(\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\frac{m-l}{2}+\frac{1}{2}\right)\Gamma(m-l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\right.\times$$

$${}_1F_2\left(\frac{2m+l}{2};\frac{2m+l}{2}+\frac{1}{2},2m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times$$

$$\left.{}_1F_2\left(\frac{m-l}{2};\frac{m-l}{2}+\frac{1}{2},m-l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=m+1}^{+\infty}\left[\frac{(-1)^l\,\Gamma\left(\frac{2m+l}{2}\right)\Gamma\left(\frac{l-m}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-m-3l}\tau^{-m}\gamma^{-l}}{2^{m+3l}\,\Gamma\left(\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\frac{l-m}{2}+\frac{1}{2}\right)\Gamma(-m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$${}_1F_2\left(\frac{2m+l}{2};\frac{2m+l}{2}+\frac{1}{2},2m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times$$

$$\left.{}_1F_2\left(\frac{l-m}{2};\frac{l-m}{2}+\frac{1}{2},-m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=1}^{2m}\left[\frac{(-1)^{m+l}\,\Gamma\left(\frac{2m-l}{2}\right)\Gamma\left(\frac{m+l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-3m-l}\tau^{-m}\gamma^{l}}{2^{3m+l}\,\Gamma\left(\frac{2m-l}{2}+\frac{1}{2}\right)\Gamma(2m-l+1)\Gamma\left(\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$${}_1F_2\left(\frac{2m-l}{2};\frac{2m-l}{2}+\frac{1}{2},2m-l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times$$

$$\left.{}_1F_2\left(\frac{m+l}{2};\frac{m+l}{2}+\frac{1}{2},m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\right]+$$

$$\sum_{m=1}^{+\infty}\sum_{l=2m+1}^{+\infty}\left[\frac{(-1)^{-m}\,\Gamma\left(\frac{l-2m}{2}\right)\Gamma\left(\frac{m+l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{m-3l}\tau^{-m}\gamma^{l}}{2^{-m+3l}\,\Gamma\left(\frac{l-2m}{2}+\frac{1}{2}\right)\Gamma(-2m+l+1)\Gamma\left(\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\right.\times$$

$$ {}_1F_2\left(\frac{l-2m}{2};\frac{l-2m}{2}+\frac{1}{2},-2m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times $$

$$ {}_1F_2\left(\frac{m+l}{2};\frac{m+l}{2}+\frac{1}{2},m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\Bigg]+ $$

$$ \sum_{m=0}^{+\infty}\sum_{l=0}^{2m}\Bigg[\frac{\Gamma\left(\frac{2m-l}{2}\right)\Gamma\left(\frac{m+l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-3m-l}\tau^{m}\,\gamma^{-l}}{2^{3m+l}\,\Gamma\left(\frac{2m-l}{2}+\frac{1}{2}\right)\Gamma(2m-l+1)\Gamma\left(\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\times $$

$$ {}_1F_2\left(\frac{2m-l}{2};\frac{2m-l}{2}+\frac{1}{2},2m-l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times $$

$$ {}_1F_2\left(\frac{m+l}{2};\frac{m+l}{2}+\frac{1}{2},m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\Bigg]+ $$

$$ \sum_{m=0}^{+\infty}\sum_{l=2m+1}^{+\infty}\Bigg[\frac{(-1)^{l}\,\Gamma\left(\frac{l-2m}{2}\right)\Gamma\left(\frac{m+l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{m-3l}\tau^{m}\,\gamma^{-l}}{2^{-m+3l}\,\Gamma\left(\frac{l-2m}{2}+\frac{1}{2}\right)\Gamma(-2m+l+1)\Gamma\left(\frac{m+l}{2}+\frac{1}{2}\right)\Gamma(m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\times $$

$$ {}_1F_2\left(\frac{l-2m}{2};\frac{l-2m}{2}+\frac{1}{2},-2m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times $$

$$ {}_1F_2\left(\frac{m+l}{2};\frac{m+l}{2}+\frac{1}{2},m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\Bigg]+ $$

$$ \sum_{l=1}^{+\infty}\sum_{m=0}^{l}\Bigg[\frac{(-1)^{m}\,\Gamma\left(\frac{2m+l}{2}\right)\Gamma\left(\frac{l-m}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-m-3l}\tau^{m}\,\gamma^{l}}{2^{m+3l}\,\Gamma\left(\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\frac{l-m}{2}+\frac{1}{2}\right)\Gamma(-m+l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\times $$

$$ {}_1F_2\left(\frac{2m+l}{2};\frac{2m+l}{2}+\frac{1}{2},2m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times $$

$$ {}_1F_2\left(\frac{l-m}{2};\frac{l-m}{2}+\frac{1}{2},-m+l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\frac{l}{2};\frac{l}{2}+\frac{1}{2},l+1;-\frac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\Bigg]+ $$

$$ \sum_{l=1}^{+\infty}\sum_{m=l+1}^{+\infty}\Bigg[\frac{(-1)^{l}\,\Gamma\left(\frac{2m+l}{2}\right)\Gamma\left(\frac{m-l}{2}\right)\Gamma\left(\frac{l}{2}\right)\left(\sqrt{a}+\sqrt{b}\right)^{-3m-l}\tau^{m}\,\gamma^{l}}{2^{3m+l}\,\Gamma\left(\frac{2m+l}{2}+\frac{1}{2}\right)\Gamma(2m+l+1)\Gamma\left(\frac{m-l}{2}+\frac{1}{2}\right)\Gamma(m-l+1)\Gamma\left(\frac{l}{2}+\frac{1}{2}\right)\Gamma(l+1)}\times $$

$$ {}_1F_2\left(\tfrac{2m+l}{2};\tfrac{2m+l}{2}+\tfrac{1}{2},2m+l+1;-\tfrac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\times $$

$$ {}_1F_2\left(\tfrac{m-l}{2};\tfrac{m-l}{2}+\tfrac{1}{2},m-l+1;-\tfrac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right){}_1F_2\left(\tfrac{l}{2};\tfrac{l}{2}+\tfrac{1}{2},l+1;-\tfrac{\left(\sqrt{a}+\sqrt{b}\right)^{-2}}{4}\right)\Bigg]\Bigg\}. \qquad (10) $$

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