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Algunos resultados sobre la subordinación diferencial de Briot-Bouquet de funciones analíticas

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Resumen

En la presente investigación se establecen algunas relaciones de subordinación para funciones analíticas pertenecientes a las clases K_0 (a,a,b) y K_1 (a,a,b), aplicando el método de subordinación diferencial de Briot-Bouquet. De lo resultados principales se discuten algunos casos especiales.

Palabras clave: Funciones analíticas, subordinaciones, subordinaciones diferenciales.

Some results on Briot-Bouquet differential subordination of analytic functions

Abstract

In the present investigation some subordination relations for functions belonging to the classes K_0 (a,a,b) and K_1 (a,a,b) are established by applying the method of Briot-Bouquet differential subordination. Some special cases of the main results are discussed.

Key words: Analytic functions, subordinations, Differential subordinations.

Introduction and definitions

Let A be the class of analytic functions f defined as

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \ , \tag{1}$$

in the open unit disc $D:=\{z\in C:\ |z|<1\}$.

A function $f\in A$ is said to be in the class $S_0^*(a)$ of starlike functions of complex order a ($a\in C\setminus\{0\}$) in D if and only if

$$\frac{f(z)}{z}\neq 0 \quad \text{and } \Re\left(1+\frac{1}{a}\left[\frac{zf'(z)}{f(z)}-1\right]\right)>0 \ \ (z\in D;\ a\in C\setminus\{0\}). \tag{2}$$

A function $f\in A$ is said to be in the class $K_0(a)$ of convex functions of complex order a ($a\in C\setminus\{0\}$) in D if and only if

$$\frac{f(z)}{z}\neq 0 \quad \text{and } \Re\left(1+\frac{1}{a}\frac{zf''(z)}{f'(z)}\right)>0 \ \ (z\in D;\ a\in C\setminus\{0\}). \tag{3}$$

This evidently leads to a conclusion that $f(z)\in K_0(a) \quad \Leftrightarrow \quad zf'(z)\in S_0^*(a)$.

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A subclass denoted by $S_1^*(a)$ contains the functions $f\in A$ satisfying the following inequality:

$$\left|\frac{zf'(z)}{f(z)}-1\right|<|a| \qquad (z\in D;\ a\in C\setminus\{0\}). \tag{4}$$

It can be noted that $S_1^*(a)$ is a subclass of $K_1(a)$.

Furthermore, $K_1(a)$ denotes a subclass of functions $f\in A$ satisfying the following inequality:

$$\left|\frac{zf''(z)}{f'(z)}\right|<|a| \qquad (z\in D;\ a\in C\setminus\{0\}), \tag{5}$$

We observe that $K_1(a)$ is a subclass of $K_0(a)$ and

$$f(z)\in K_1(a) \quad \Leftrightarrow \quad zf'(z)\in S_1^*(a) \qquad (z\in D;\ a\in C\setminus\{0\}).$$

If we set $a=1-\alpha$ ($0\le\alpha<1$), we get

$$S_0^*(1-\alpha)=S^*(\alpha) \text{ and } K_0(1-\alpha)=K(\alpha), \tag{6}$$

Where $S^*(\alpha)$ and $K(\alpha)$ denote familiar classes of starlike and convex functions of a real order α in D.

The classes $S_0^*(a)$ and $K_0(a)$ of starlike and convex functions of a complex order a in D were introduced and investigated earlier by Nasr and Aouf [6] and Wiatrowski [10] and many others. Their subclasses $S_1^*(a)$ and $K_1(a)$ were studied by (among others) Choi [1] and Lashin [4]. Recently, Srivastava and Lashin [9] studied the starlike and convex functions of complex order. Frasin and Darus [3] have defined a class $B(\alpha)$ and investigated some interesting properties for this class. Siregar et al. [7] studied a subclass $B_a^*(\alpha)$ of the class A and derived some subordination and superordination relations.

We now define for $0\le\alpha<1$, $b\in N$ following subclasses of A

$$K_0(\alpha,a,b)=\left\{ f\in A:\ \Re\left\{\left(1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1 \right) \right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-\left(\frac{bzf'(z)}{f(z)}-b\right)\right)\right\}>\alpha;\ z\in D\right\}$$
$$\left(a\in C\setminus\{0\}\right) \tag{7}$$

and

$$K_1(\alpha,a,b)=\left\{ f\in A:\ \left|\left(\frac{zf''(z)}{f'(z)}+\left(\frac{z^b f'(z)}{f^b(z)}-1\right)-b\left(\frac{zf'(z)}{f(z)}-1\right)\right)\right|<|a|(1-\alpha);\ z\in D\right\};\ a\in C\setminus\{0\}. \tag{8}$$

It is interesting to note that,

$$K_0(0,a,1)=K_0(a),\quad K_1(0,a,1)=K_1(a),\quad \text{and } K_0(0,1-\alpha,1)=K(\alpha);\ 0\le\alpha<1. \tag{9}$$

$$K_0(\alpha,a,2)=B_a^*(\alpha'),\quad K_0(\alpha-\alpha',a,2)=B_a^*,\ \text{ where } \alpha'=\Re\left(\alpha-1+\frac{1}{a}\right), \tag{10}$$

here $K_0(a)$ and $K_1(a)$ are defined by (3) and (5), and B_a^* and $B_a^*(\alpha)$ are subclasses investigated in [8]. It is interesting to note the following

$$\alpha_1<\alpha_2 \quad \text{then} \quad K_0(\alpha_2,a,b)\subset K_0(\alpha_1,a,b), \tag{11}$$

also
$$f\in K_1(\alpha,a,b)\ \Rightarrow\ f\in K_0(\alpha,a,b).$$

In the present work, we investigate certain relations of subordination for analytic functions belonging to the newly defined classes. The method of Briot-Bouquet differential subordination is used to derive the results.

We shall need the following definitions:

Definition 1. Let the functions $f(z)$ and $g(z)$ be analytic in D. The function $f(z)$ is said to be subordinate to the function $g(z)$, written symbolically as

$$f(z)\prec g(z)\quad (z\in D),$$

if there exists a function $w(z)$ analytic in D with

$$w(0)=0 \text{ and } |w(z)|<1\quad (z\in D),$$

such that
$$f(z)=g(w(z)),\quad (z\in D)$$

Furthermore, if the function $g(z)$ is univalent in D then

$$f(z)\prec g(z)\ (z\in D)\ \Leftrightarrow\ f(0)=g(0) \text{ and } f(D)\subseteq g(D).$$

Definition 2. Let $\varphi: C^2\to C$ be an analytic function and let $h(z)$ be univalent in D. If $p(z)$ is analytic in D, then $p(z)$ is called a solution of the differential subordination, when it satisfies the differential subordination

$$\varphi\left(p(z),\ zp'(z)\right)\prec h(z)\quad (z\in D). \tag{12}$$

Definition 3. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (12), if

$$p(z) \prec q(z) \qquad (z \in D), \tag{13}$$

for all functions $p(z)$ satisfying the subordination (12).

Moreover, if $\tilde{q}(z) \prec q(z) \qquad (z \in D),$

for all dominants of (13), then we say that $\tilde{q}(z)$ is the best dominant of (13).

Preliminaries

We first mention the known results required in the present study.

Lemma1. Let the function $w(z)$ be analytic and convex in D, then the function defined as

$$h(z) = z + w(z) \tag{14}$$

is also convex in D.

Lemma2. (cf. [5], p. 17 et seq.) Let the functions $f(z)$ and $g(z)$ be analytic in the unit disk D and let

$$f(0) = g(0).$$

If the function H (z) := $z\,g'(z)$ is starlike in D and

$$zf'(z) \prec zg'(z),$$

then

$$f(z) \prec g(z) = g(0) + \int_0^z \frac{H(t)}{t}\,dt, \tag{15}$$

The function $g(z)$ is convex and is the best dominant in (15).

Lemma3. ([2]) Let β and γ be complex constants. Also let the function $h(z)$ be convex (univalent) in D with

$$h(0) = 1 \text{ and } \Re(\beta h(z) + \gamma) > 0,\ z \in D.$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots ,$$

is analytic in D and satisfies the following differential subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{16}$$

If the differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (q(0) := 1), \tag{17}$$

has a univalent solution $q(z)$, then

$$p(z) \prec q(z) \prec h(z)$$

and $q(z)$ is the best dominant in (17) .

Remark1. The conclusion of Lemma 3 can be written in the following form:

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec q(z)+\frac{zq'(z)}{\beta q(z)+\gamma} \Rightarrow p(z)\prec q(z).$$

Remark2. The solution of the differential equation (17) is given by

$$q(z)=\frac{zF'(z)}{F(z)}=\frac{\beta+\gamma}{\beta}\left(\frac{H(z)}{F(z)}\right)^{\beta}-\frac{\gamma}{\beta} \quad (\beta\neq 0),$$

where
$$F(z)=\left(\frac{\beta+\gamma}{z^{\gamma}}\int_0^z \{H(t)\}^{\beta} t^{\gamma-1}\,dt\right)^{\frac{1}{\beta}}$$

and
$$H(z)=z.exp\left(\int_0^z \frac{h(t)-1}{t}dt\right).$$

Main Subordination Results

Theorem1. Let $f(z)\in A$ and $h(z)$ be a convex univalent function in D such that

$$h(0)=1 \quad \text{and} \quad \Re(ah(z)+(1-a))>0;\quad z\in D;\quad a\in C\setminus\{0\}.$$

(a) If

$$\left(1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right)\right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-b\left(\frac{zf'(z)}{f(z)}-1\right)\right)\prec h(z) \quad (b\in N) \tag{18}$$

then

$$1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right)\prec h(z). \tag{19}$$

(b) If the following differential equation:

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma} = h(z) \qquad (q(0):=1),$$

has a univalent solution $q(z)$, then

$$\left(1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right)\right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-b\left(\frac{zf'(z)}{f(z)}-1\right)\right)\prec h(z)$$

$$\Rightarrow 1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right)\prec q(z)\prec h(z) \tag{20}$$

and $q(z)$ is the best dominant in (20).

Proof . Let $$1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right) =: p(z), \qquad (21)$$

so that $p(z)$ has the following expansion:

$$p(z)=1+p_1 z + p_2 z^2 + \ldots \ . \qquad (22)$$

Logarithmic differentiation of (21) gives

$$p(z)+\frac{zp'(z)}{ap(z)+(1-a)} = 1+\frac{1}{a}\left(\frac{zbf'(z)}{f^b(z)}-1\right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-b\left(\frac{zf'(z)}{f(z)}-1\right)\right) .$$

Thus (18) is expressed in subordination form as follows

$$p(z)+\frac{zp'(z)}{ap(z)+(1-a)} \prec h(z).$$

In view of Lemma 3 if β and γ are replaced by a and $(1- a)$ respectively, the conclusions of the theorem immediately follow. This completes the proof.

Remark3. If we take $b = 1$ and $\alpha = 0$ then the above result reduces to Theorem 1 obtained in [9].

Setting $b = 2$ and considering the subclass given in (10) then the above result reduces to Theorem 3.1 obtained in [7].

Our second assertion is as follows:

Theorem2. Let $f(z) \in K_1(\alpha, a, b)$ then for $\Re(1+a(1-\alpha)z)>0$ $z \in D$; $|a| \le 1$; $a \ne 0$,

$$1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right) \prec q(z), \qquad (23)$$

where $q(z)$ is the best dominant given by

$$q(z)=1-\frac{1}{a}+\frac{(1-\alpha)ze^{a(1-\alpha)z}}{\left(e^{a(1-\alpha)z}-1\right)} \ . \qquad (24)$$

Proof. When $f(z) \in K_1(\alpha, a, b)$, then from (8) we get

$$\left| \left(1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right) \right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-b\left(\frac{zf'(z)}{f(z)}-1\right)\right)-1 \right| < (1-\alpha), \quad z \in D.$$

In other words

$$\left(1+\frac{1}{a}\left(\frac{z^b f'(z)}{f^b(z)}-1\right) \right)+\frac{1}{a}\left(\frac{zf''(z)}{f'(z)}-b\left(\frac{zf'(z)}{f(z)}-1\right)\right) \prec 1+(1-\alpha)z, \quad z \in D. \qquad (25)$$

In Theorem 1 if we take

$$h(z)=1+(1-\alpha)z \qquad (z\in D)$$

and assume that

$$\Re(ah(z)+(1-a))>0,\ |a|\le 1,\ a\ne 0,$$

then by applying Lemma 3 and Remark 2, solution of the differential subordination (25) is given by the relation (23), where $q(z)$ is calculated as below

$$q(z)=\frac{1}{a}\left(\frac{H(z)}{F(z)}\right)^{a}-\frac{1-a}{a},$$

for $$H(z)=z.e^{(1-\alpha)z}$$ and

$$F(z)=\left(\frac{1}{a(1-\alpha)z^{1-a}}\left(e^{a(1-\alpha)z}-1\right)\right)^{\frac{1}{a}}.$$

This completes the proof of the theorem.

Remark4. If we take b = 1 and $\alpha=0$ then the above result reduces to Theorem 2 obtained in [9].

If we set b = 2 in our Theorem 2 then it reduces to the following:

Corollary1. Let $f(z)\in K_1(\alpha,a,2)$ then for $\Re(1+a(1-\alpha)z)>0$; $a\ne 0$, $z\in D$;

$$\frac{z^2 f'(z)}{a f^2(z)} \prec \frac{(1-\alpha)ze^{a(1-\alpha)z}}{\left(e^{a(1-\alpha)z}-1\right)}.$$

In Theorem 2 when α = 1 and $\alpha=0$ we get:

Corollary2. Let $f(z)\in K_1(0,1,b)$ then for $z\in D$,

$$\frac{z^b f'(z)}{f^b(z)} \prec q(z), \tag{26}$$

where $q(z)$ is the best dominant given by $q(z)=\dfrac{ze^z}{\left(e^z-1\right)}$.

Further, taking b=1 and simplifying the subordination result with the help of the Lemma 1 and 2, we obtain:

Corollary3. Let $f(z)\in K_1(0,1,1)$ then for $z\in D$,

$$f(z)\prec\left(e^z-1\right). \tag{27}$$

Proof. Since $f(z)\in K_1\ (0,1,1)$, then from Corollary2

$$\frac{z\ f'(z)}{f\ (z)} \prec \frac{ze^z}{(e^z-1)} ,$$

the above relation can be re expressed as

$$zG'(z) \prec zH'(z),$$

where $$G(z)=log\ f(z) \text{ and } H(z)=log(e^z-1).$$

It is well known that the Bernoulli function $h(z)=\frac{z}{(e^z-1)}$ is an analytic and convex function in D (*c.f.* [5] Theorem A3, pp.417) thus with the help of Lemma 1, it is deduced that $zH'(z)=\frac{z\,e^z}{e^z-1}$, is a starlike function. Applying the Lemma2, we obtain

$$G(z) \prec H(z).$$

This evidently leads to the assertion (27).

From Corollary3, and relations (6), (8) and (9) we deduce for $\alpha=0$ that

Corollary4. Let $f(z)\in S^*$. If $\left|\frac{zf'(z)}{f(z)}-1\right|<1,$ for $z\in D$, then

$$\int_0^z \frac{f(t)}{t}dt \prec (e^z-1).$$

Remark5. If we consider the result ([8], Theorem1, p. 203) , it can easily be deduced that, if $f(z)\in S_0^*(a)$, $a\in C\setminus\{0\}$, then

$$\left(\frac{f(z)}{z}\right)^b \prec \frac{1}{(1-z)^{2ab}}, \qquad (28)$$

provided that $b\neq 0$ and $|ab|\le 1$ and the function $\frac{1}{(1-z)^{2ab}}$ is the best dominant of the subordination (28).

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