Revista Tecnocientífica URU
Universidad Rafael Urdaneta
Facultad de Ingeniería
Nº 6 Enero - Junio 2014
Depósito legal: ppi 201402ZU4464
ISSN: 2343 - 6360

Una nueva clase de polinomios *q*-Apostol-Bernoulli de orden *α*

Mridula Garg and Subhash Alha

Department of Mathematics, University of Rajasthan, Jaipur 302004, Rajasthan, India
gargmridula@gmail.com and subhashalha@gmail.com

Recibido: 03-10-2013 Aceptado: 22-03-2014

Resumen

En este trabajo, en primer lugar se da una introducción de los números y polinomios de Bernoulli y sus *q*-generalizaciones. Luego se define una nueva clase de polinomios *q*-Apostol-Bernoulli de orden *α* y sus números correspondientes. Se obtienen representaciones explicitas, teorema de adición y fórmula diferenciales de esta nueva clase de polinomios.

Palabras clave: Polinomios *q*-Apostol-Bernoulli

A new class of *q*-Apostol-Bernoulli polynomials of order *α*

Abstract

In this paper, we first give an introduction of Bernoulli polynomials and numbers and their *q*-generalizations. We then define a new class of *q*-Apostol-Bernoulli polynomials of order *α* and corresponding numbers. We obtain explicit representations, addition theorem and differential formula for these newly defined class of polynomials.

Key words: *q*-Apostol-Bernoulli polynomials

Notations and Definitions

We shall use the following notations and definitions of *q*-theory (Gasper & Rahman [8])

The *q*-number $[x]_q$ and the *q*-number factorial $[n]_q!,\ n \in N$ are defined by

$$[x]_q = \frac{1-q^x}{1-q} \quad q \neq 1. \text{ and } [n]_q! = \prod_{j=1}^{n}[j]_q. \tag{1}$$

The *q*-shifted factorial (*q*-analogue of Pochhammer symbol) is defined as

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right),\ n \in N \text{ with } (a;q)_0 = 1,\ q \neq 1. \tag{2}$$

If we consider $(a;q)_\infty$ then as the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, therefore whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

Further, for any complex number α, we have

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{3}$$

There is one more definition of q-number shifted factorial,which is often used in the definitions of q-extension of Bernoulli polynomials. This is as follows

$$[a]_{q;n} = \prod_{j=0}^{n-1}[a+j]_q \text{, with } [a]_{q;n} = (1-q)^{-n}(q^a;q)_n. \tag{4}$$

The q-binomial theorem is given by

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad |z|<1,\ 0<|q|<1. \tag{5}$$

For $a = q^\alpha;\ (\alpha \in C)$ the result can be written as follows

$$\sum_{n=0}^{\infty}\frac{(q^\alpha;q)_n}{(q;q)_n}z^n = \frac{(q^\alpha z;q)_\infty}{(z;q)_\infty} = \frac{1}{(z;q)_\alpha} \qquad |z|<1,\ 0<|q|<1. \tag{6}$$

Introduction

The definitions of classical Bernoulli polynomials $B_n(x)$ and numbers B_n and their familiar generalizations $B_n^{(\alpha)}(x)$ and $B_n^{(\alpha)}$, Bernoulli polynomials and numbers of order α, can be seen in the texts (Erdélyi et al. [6]; Olver et al.[21]). Some interesting analogues of the classical Bernoulli polynomials were investigated by Apostol [1], so-called Apostol Bernoulli polynomials $B_n(x;\lambda)$. Further, Luo and Srivastava [18] introduced and investigated the Apostol-Bernoulli polynomials of order α, $B_n^{(\alpha)}(x;\lambda)$. Some more generalizations and analogues of these polynomials have been studied by researchers namely Natalini and Benardini[20], Luo et al.[14], Breeti[2], Srivastava et al. [23],Kurt [12], Tremblay [25].

q-analogues of Bernoulli numbers were first studied by Carlitz[3].Thereafter various other q-analogues of Bernoulli numbers and polynomials have been studied arising from varying motivations. Many authors have further studied and developed this subject, among which a few to mention are Koblitz [11], Tsumura [26], Srivastava et al. [24], Cenkciand Can[4],Ernst [7],Ryoo [22], Choi et al. [5], Kim et al. [10], Luo [17], Luo and Srivastava [15], Mahmudov [19] and Lee and Ryoo[13]. We recall here some of these definitions.

q-extensions of Bernoulli polynomials and numbers of order $\alpha \in C$, are defined by means of the following generating functions (see Luo and Srivastava [15])

$$(-z)^\alpha \sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^{n+x}e^{[n+x]_q z} = \sum_{n=0}^{\infty}B_{n;q}^{(\alpha)}(x)\frac{z^n}{n!}, \tag{7}$$

$$\left(-z\right)^{\alpha}\sum_{n=0}^{\infty}\frac{\left[\alpha\right]_{q;n}}{\left[n\right]_{q}!}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}B_{n;q}^{(\alpha)}\frac{z^{n}}{n!}. \tag{8}$$

The following formula for $B_{n;q}^{(\alpha)}\left(x\right)$ in terms of $B_{j;q}^{(\alpha)}$ can easily be obtained from (7) and (8)

$$B_{n;q}^{(\alpha)}\left(x\right)=\sum_{j=0}^{n}\binom{n}{j}\{[x]_{q}\}^{n-j}q^{(j+1-\alpha)x}B_{j,q}^{(\alpha)}. \tag{9}$$

Remark 1. *From the relation (9), it is obvious that the degree of* $B_{n,q}^{(\alpha)}\left(x\right)$ *is* $\left(n+1-\alpha\right)$ *in* q^{x}*, which means that for non integral values of* α*, it is not a polynomial in* q^{x}*.*

Cenkci and Can [4] introduced ***q***-extensions of Apostol-Bernoulli polynomials and numbers are defined by means of the following generating functions

$$\left(-z\right)\sum_{n=0}^{\infty}\lambda^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}\left(x;\lambda;q\right)\frac{z^{n}}{n!}, \tag{10}$$

$$\left(-z\right)\sum_{n=0}^{\infty}\lambda^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}\left(\lambda;q\right)\frac{z^{n}}{n!} \tag{11}$$

Choi et al. [5] gave the following definitions for q-extensions of Apostol-Bernoulli polynomials and numbers of order $k\in N$,

$$\left(-z\right)^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\lambda^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)\frac{z^{n}}{n!}, \tag{12}$$

$$\left(-z\right)^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\lambda^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathfrak{B}_{n}^{(k)}\left(\lambda;q\right)\frac{z^{n}}{n!}. \tag{13}$$

The following formula can easily be obtained from (12) and (13)

$$\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)=\sum_{j=0}^{n}\binom{n}{j}\{[x]_{q}\}^{n-j}q^{(j+1-k)x}\mathfrak{B}_{j}^{(k)}\left(\lambda;q\right) \tag{14}$$

Remark 2. *It is observed from (14), that the degree of* $\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)$ *is* $\left(n+1-k\right)$ *in* q^{x}*, whereas, the notation* $\mathfrak{B}_{n}^{(k)}\left(x;\lambda;q\right)$ *indicates that it should be of degree n.*

We would like to mention here that in the same paper the authors have also defined a q-extension of Apostol-Euler polynomials and numbers of order $k\in N$ by the following generating functions

$$2^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\left(-\lambda\right)^{n}q^{n+x}e^{[n+x]_{q}z}=\sum_{n=0}^{\infty}\mathscr{E}_{n}^{(k)}\left(x;\lambda;q\right)\frac{z^{n}}{n!} \tag{15}$$

$$2^{k}\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_{q}!}\left(-\lambda\right)^{n}q^{n}e^{[n]_{q}z}=\sum_{n=0}^{\infty}\mathscr{E}_{n}^{(k)}\left(\lambda;q\right)\frac{z^{n}}{n!} \tag{16}$$

From equations (3.18) and (3.32) of the same paper (Choi et al. [5]), it is easy to derive the following relation between $\mathfrak{B}_{n}^{(k)}\left(\lambda;q\right)$ and $\mathscr{E}_{n}^{(k)}\left(\lambda;q\right)$, which shows that these are not independent

$$\mathfrak{B}_n^{(k)}\left(-\lambda;q\right)=\left(-1\right)^n\left(k+1\right)_n\mathscr{E}_n^{(k)}\left(\lambda;q\right)\qquad\left(k\in N_0,\ n\in N\right)\tag{17}$$

In the present paper, we further extend this study and define a q-extension of Apostol-Bernoulli polynomials of order $B_{n,q}^{(\alpha,\lambda)}(x)$ $\alpha,\lambda\in C$. We shall also prove in Theorem 1 that these are polynomials of degree n in q^x. This property overcomes the shortcoming pointed out in Remark 2 of earlier definition $\mathfrak{B}_n^{(k)}\left(x;\lambda;q\right)$.

A new class of q-Apostol-Bernoulli polynomials and numbers of order α

Definition. For α, $\lambda\in C$, $0<\left|q\right|<1$, we define a new class of q-Apostol-Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,\lambda)}(x)$ by means of the following generating function

$$\frac{1}{\left(-1\right)^{\alpha}}\sum_{n=0}^{\infty}\frac{\left[\alpha\right]_{q;n}}{\left[n\right]_q!}\lambda^n q^n e^{[n+x]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x)\frac{z^n}{n!}\tag{18}$$

and corresponding numbers are given by

$$\frac{1}{\left(-1\right)^{\alpha}}\sum_{n=0}^{\infty}\frac{\left[\alpha\right]_{q;n}}{\left[n\right]_q!}\lambda^n q^n e^{[n]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}\frac{z^n}{n!}.\tag{19}$$

Obviously, $B_{n,q}^{(\alpha,\lambda)}=B_{n,q}^{(\alpha,\lambda)}(0)$. (20)

Special Cases

1. If we set $\alpha=k\in N$ in (18) and (19), we get

$$\left(-1\right)^k\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_q!}\lambda^n q^n e^{[n+x]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(k,\lambda)}(x)\frac{z^n}{n!},\tag{21}$$

$$\left(-1\right)^k\sum_{n=0}^{\infty}\frac{\left[k\right]_{q;n}}{\left[n\right]_q!}\lambda^n q^n e^{[n]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(k,\lambda)}\frac{z^n}{n!}.\tag{22}$$

We observe that $B_{n,q}^{(k,\lambda)}(x)$ is associated with $\mathfrak{B}_n^{(k)}\left(x;\lambda;q\right)$ given by (12) according with the following relation

$$B_{n,q}^{(k,\lambda)}(x)=\frac{n!}{q^x\left(n+k\right)!}\mathfrak{B}_{n+k}^{(k)}\left(x;\lambda;q\right).\tag{23}$$

Further on taking $k=1$ in (21) and (22), we arrive at the following q-extension of Apostol-Bernoulli polynomials and numbers

$$\left(-1\right)\sum_{n=0}^{\infty}\lambda^n q^n e^{[n+x]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(1,\lambda)}(x)\frac{z^n}{n!},\tag{24}$$

$$\left(-1\right)\sum_{n=0}^{\infty}\lambda^n q^n e^{[n]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(1,\lambda)}\frac{z^n}{n!}.\tag{25}$$

Here, $B_{n,q}^{(1,\lambda)}(x)$ is associated with $B_n\left(x;\lambda;q\right)$ given by (10), according with the following relation

$$B_{n,q}^{(1,\lambda)}(x) = \frac{1}{q^x(n+1)} B_{n+1}(x;\lambda;q). \tag{26}$$

2. If we take $q \to 1$ in (18) and (19), we get the following class of Bernoulli polynomials $B_n^{(\alpha,\lambda)}(x)$ and numbers $B_n^{(\alpha,\lambda)}$

$$\frac{1}{(\lambda e^z - 1)^\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha,\lambda)}(x)\frac{z^n}{n!} \qquad (|z| < \ln\lambda) \tag{27}$$

$$\frac{1}{(\lambda e^z - 1)^\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha,\lambda)} \frac{z^n}{n!} \qquad (|z| < \ln\lambda). \tag{28}$$

We observe that $B_n^{(\alpha,\lambda)}(x)$ is not comparable with the definition of Apostol-Bernoulli polynomials of order α, $B_n^{(\alpha)}(x;\lambda)$ defined by Luo and Srivastava [18] as follows

$$\left(\frac{z}{\lambda e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}. \tag{29}$$

Rather it is related with Apostol Euler polynomials of order α, $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ [18] through the following relation

$$B_n^{(\alpha,-\lambda)}(x) = \frac{1}{(-2)^\alpha} \mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{30}$$

3. If we set $\lambda = 1$ in (18) and (19), we get the following q-extension of Bernoulli polynomials and numbers of order α

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} q^n e^{[n+x]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,1)}(x)\frac{z^n}{n!}, \tag{31}$$

$$\frac{1}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} q^n e^{[n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,1)} \frac{z^n}{n!}. \tag{32}$$

Here, $B_{n,q}^{(\alpha,1)}(x)$ is not comparable with the definitions $B_{n,q}^{(\alpha)}(x)$ given by (7) but we have the relation between $B_{n,q}^{(\alpha,1)}(x)$ and q-Euler polynomials of order α, $E_{n,q}^{(\alpha)}(x)$ (see [Luo and Srivastava [15]) by the following relation

$$B_{n,q}^{(\alpha,1)}(x) = \frac{(-1)^n}{(-2)^\alpha q^x} E_{n,q}^{(\alpha)}(x). \tag{33}$$

We would like to remark here that $B_{n,q}^{(\alpha,1)}(x)$ given by (31) has an improvement over $B_{n,q}^{(\alpha)}(x)$ given by (7) in the sense that for non integral values of α, it is polynomial of degree n in q^x as obvious from the relation (34).

Explicit Representations

Theorem 1. For $\alpha, \lambda \in C, 0 < |q| < 1$, we have

(a) $$B_{n,q}^{(\alpha,\lambda)}(x) = \sum_{j=0}^{n} \binom{n}{j} \{[x]_q\}^{n-j} q^{jx} B_{j,q}^{(\alpha,\lambda)}, \tag{34}$$

(b) $$B_{n,q}^{(\alpha,\lambda)} = \frac{1}{(-1)^\alpha} \sum_{j=0}^{\infty} \frac{[\alpha]_{q;n}}{[j]_q!} q^j \lambda^j \left([j]_q\right)^n, \tag{35}$$

Where $B_{n,q}^{(\alpha,\lambda)}(x)$ and $B_{n,q}^{(\alpha,\lambda)}$ are defined by equation (18) and (19). It is clear from (34) that $B_{n,q}^{(\alpha,\lambda)}(x)$ is a polynomial of degree n in q^x.

Proof (a). Using the relation $[x+n]_q = [x]_q + q^x[n]_q$, we can write (18) as

$$\frac{e^{[x]_q z}}{(-1)^\alpha} \sum_{n=0}^{\infty} \frac{[\alpha]_{q;n}}{[n]_q!} \lambda^n q^n e^{q^x [n]_q z} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{36}$$

Using (19) in L.H.S. of (36), we get

$$e^{[x]_q z} \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}.q^{nx}.\frac{z^n}{n!} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{37}$$

Writing series for $e^{[x]_q z}$ we have the following

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}.q^{nx}.\frac{\left([x]_q\right)^j}{j!} \frac{z^{n+j}}{n!} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha,\lambda)}(x) \frac{z^n}{n!}. \tag{38}$$

Using series manipulation and equating coefficients of $\frac{z^n}{n!}$ we get the desired result (34).

(b). Result (35) can easily be obtained from (19) on using series of exponential function and equating coefficients of $\frac{z^n}{n!}$.

Form equation (34), it is easy to see that $B_{n,q}^{(\alpha,\lambda)}(x)$ is a polynomial of degree n in q^x.

We now calculate the values of q-Apostol-Bernoulli numbers $B_{n,q}^{(\alpha,\lambda)}$ and polynomials $B_{n,q}^{(\alpha,\lambda)}(x)$ for different values of n with the help of (35) and (34).

Few q-Apostol-Bernoulli numbers as calculated from (35) are

$$\left.\begin{aligned}
B_{0,q}^{(\alpha,\lambda)} &= \frac{1}{(-1)^{\alpha}}\frac{1}{(q\lambda;q)_{\alpha}},\\
B_{1,q}^{(\alpha,\lambda)} &= \frac{q\lambda}{(-1)^{\alpha}}\frac{[\alpha]_q}{(q\lambda;q)_{\alpha+1}},\\
B_{2,q}^{(\alpha,\lambda)} &= \frac{q^2\lambda^2}{(-1)^{\alpha}}\frac{[\alpha]_q[\alpha+1]_q}{(q\lambda;q)_{\alpha+2}}+\frac{q\lambda}{(-1)^{\alpha}}\frac{[\alpha]_q}{(q^2\lambda;q)_{\alpha+1}},\dots
\end{aligned}\right\} \tag{39}$$

Further using above values in (34), we get few q-Apostol-Bernoulli polynomials as follows

$$\left.\begin{aligned}
B_{0,q}^{(\alpha,\lambda)}(x) &= B_{0,q}^{(\alpha,\lambda)}\\
B_{1,q}^{(\alpha,\lambda)}(x) &= \frac{1}{1-q}B_{0,q}^{(\alpha,\lambda)}-\frac{q^x}{1-q}\left[B_{0,q}^{(\alpha,\lambda)}-(1-q)B_{1,q}^{(\alpha,\lambda)}\right]\\
B_{1,q}^{(\alpha,\lambda)}(x) &= \frac{1}{1-q}B_{0,q}^{(\alpha,\lambda)}-\frac{2q^x}{1-q}\left[B_{0,q}^{(\alpha,\lambda)}-B_{1,q}^{(\alpha,\lambda)}\right]+\frac{q^{2x}}{1-q}\left[B_{0,q}^{(\alpha,\lambda)}-2B_{1,q}^{(\alpha,\lambda)}+(1-q)B_{2,q}^{(\alpha,\lambda)}\right]
\end{aligned}\right\} \tag{40}$$

Theorem 2. For $\alpha,\ \lambda \in C$, $0 < |q| < 1$, we have

$$B_{n,q}^{(\alpha,\lambda)}(x) = \frac{1}{(-1)^{\alpha}}\frac{1}{q^x}\Phi_{\alpha,q}(\lambda,-n,x). \tag{41}$$

where $\Phi_{\mu,q}(z,s,a)$ (see Choi et al.[5]) is a q-extension of generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$ defined by Goyal and Laddha[9] and is defined as follows

$$\Phi_{\mu,q}(z,s,a) = \sum_{n=0}^{\infty}\frac{[\mu]_{q;n}}{[n]_q!}\frac{q^{n+a}}{\left([n+a]_q\right)^s}z^n \qquad \left(\mu,s\in C;\operatorname{Re}(a)>0\right). \tag{42}$$

Proof. In (18), if we write exponential function $e^{[n+x]_q z}$ in series form and compare it with (42) we easily arrive at (41).

If we let $q \to 1$ in (41), we get the following explicit representation for $B^{(\alpha\ \lambda)}(x)$

$$B_{n}^{(\alpha,\lambda)}(x) = \frac{1}{(-1)^{\alpha}}\Phi_{\alpha}^{*}(\lambda,-n,x) \qquad \left(|\lambda|<1\right). \tag{43}$$

Addition Theorem

Theorem 3. For $\alpha,\ \lambda \in C$, $0 < |q| < 1$, we have

$$B_{n,q}^{(\alpha,\lambda)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\{[y]_q\}^{n-j}q^{jy}B_{j,q}^{(\alpha,\lambda)}(x) \tag{44}$$

Proof. It follows from (18) that

$$\frac{1}{(-1)^{\alpha}}\sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^n\lambda^n e^{[n+x+y]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{45}$$

Using the relation $[x+n+y]_q=[y]_q+q^y[x+n]_q$, (45) can be written as

$$\frac{e^{[y]_q z}}{(-1)^{\alpha}}\sum_{n=0}^{\infty}\frac{[\alpha]_{q;n}}{[n]_q!}q^n\lambda^n e^{q^y[n+x]_q z}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{46}$$

Using (18), the above result assumes the following form

$$e^{[y]_q z}\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x)q^{ny}\frac{z^n}{n!}=\sum_{n=0}^{\infty}B_{n,q}^{(\alpha,\lambda)}(x+y)\frac{z^n}{n!}. \tag{47}$$

Expanding $e^{[y]_q z}$ in series form, using series manipulation and equating coefficients of $\frac{z^n}{n!}$, we get the result (44).

If we let $q\to 1$ in (44) we get the following addition formula for $B_n^{(\alpha,\lambda)}(x)$, defined by (27)

$$B_n^{(\alpha,\lambda)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha,\lambda)}(x)y^{n-k}. \tag{48}$$

If we take $\lambda=1$ in (44), we get the following result for q-extension of Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,1)}(x)$ defined by (31).

$$B_{n,q}^{(\alpha,1)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}\{[y]_q\}^{n-j}q^{jy}B_{j,q}^{(\alpha,1)}(x). \tag{49}$$

Differential Formula

Thoerem 4. For $\alpha, \lambda\in C$, $0<|q|<1$, we have

$$\frac{d}{dx}B_{n,q}^{(\alpha,\lambda)}(x)=n\frac{q^x\ln q}{(q-1)}B_{n-1,q}^{(\alpha,\lambda q)}(x). \tag{50}$$

Proof. Differentiating the generating function (18) w.r.t. x and using the following result

$$\frac{d}{dx}\left\{e^{q^n[x]_q z}\right\}=\frac{q^{n+x}\ln q}{q-1}ze^{q^n[x]_q z}, \tag{51}$$

we easily arrive at (50).

If we take the limit $q\to 1$ in (50), it gives the following differential formula for $B_n^{(\alpha,\lambda)}(x)$ defined by (27).

$$\frac{d}{dx}B_n^{(\alpha,\lambda)}(x)=nB_{n-1}^{(\alpha,\lambda)}(x). \tag{52}$$

If we take $\lambda = 1$ in (50), we get the following result for q-extension of Bernoulli polynomials of order α, $B_{n,q}^{(\alpha,1)}(x)$ defined by (31).

$$\frac{d}{dx} B_{n,q}^{(\alpha,1)}(x) = n \frac{q^x \ln q}{(q-1)} B_{n-1,q}^{(\alpha,q)}(x). \tag{53}$$

Acknowledgement

The support provided by the University Grants Commission, New Delhi through Rajiv Gandhi National Fellowship to one of the authors Subhash Alha is gratefully acknowledged.

References

1. Apostol, T.M.,On the Lerch Zeta function, Pacific J. Math., 1 (1951), 161-167.
2. Bretti, G, Natalini, P, Ricci, P.E., Generalizations of the Bernoulli and Appell polynomials. Abstr. Appl. Anal., 7(2004), 613-623.
3. Carlitz, L., q-Bernoulli numbers and polynomials, Duke Math. J., 15 (1948) 987-1000.
4. Cenkci, M., Can, M.,Some results on q-analogue of the Lerch zeta function, Adv. Stud. Contemp. Math., 12 (2006), 213-223.
5. Choi, J., Anderson, P.J., Srivastava, H.M.,Some q-extensions of the Apostol-Bernoulli and the Apostol–Euler polynomials of order n and the multiple Hurwitz zeta function, Appl. Math. Comput., 199 (2008), 723-737.
6. Erdélyi, A., Magnus, W., Oberhittenger, F, Tricomi, F. G., Higher Transcendental Functions, Vol. 1. McGraw-Hill, New York, Toronto and London (1953).
7. Ernst, T., q-Bernoulli and q-Stirling polynomials, an umbral approach, Int. J. Difference Equ., 1(2006), 31-80.
8. Gasper, G., Rahman, M., Basic Hypergeometric Series, Cambridge University Press, Cambridge (2004).
9. Goyal, S.P., Laddha, R.K.,On the generalized Riemann Zeta functions and the generalized Lambert transform, Ganita Sandesh, 11(1997), 99–108.
10. Kim, T., Kim, Y.-H. and Hwang, K.-W, On the q-extensions of the Bernoulli and Euler numbers related identities and Lerch zeta function, Proc. Jangjeon Math. Soc., 12 (2009), 77-92.
11. Koblitz, N., On Carlitz's q-Bernoulli numbers, J. Number Theory, 14 (1982), 332-339.
12. Kurt, B., A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x,y)$, Appl. Math. Sci., 4(47) (2010), 2315-2322.
13. Lee, H.Y., Ryoo, C.S., A note on the generalized higher-order q-Bernoulli numbers and polynomials with weight α, Taiwanese J. Math., 17(3) (2013), 785-800.
14. Luo, Q. M., Guo, B. N., Qi, F., Debnath, L.,Generalizations of Bernoulli numbers and polynomials, Int. J. Math. Math. Sci., 59 (2003), 3769-3776.
15. Luo, Q.-M., Srivastava, H.M., q-Extensions of some relationship between the Bernoulli and Euler polynomials, Taiwanese J. Math., 15(1) (2011), 241-257.
16. Luo, Q.-M., Apostol-Euler polynomials of higher order and Gaussian hypergeometric functioins, Taiwanese J. Math., 10 (2006), 917-925.

17. Luo, Q.-M., Some results for the q-Bernoulli polynomials and q-Euler polynomials, J. Math. Anal. Appl., 363 (2010), 7-18.

18. Luo, Q.-M., Srivastava, H. M., Some generalizations of the Apostol-Bernoulli and Apostol-Euler Polynomials, J. Math. Anal. Appl., 308(2005), 290–302.

19. Mahmudov, N. I., On a class of q-Bernoulli and q-Euler polynomials. Adv. Differ. Equ., 2013:108 (2013),doi:10.1186/1687-1847-2013-108.

20. Natalini, P., Bernardini, A., A generalization of the Bernoulli Polynomials, J. Appl. Math., 3 (2003), 155–163.

21. Olver F.W.J., Lozier D.W., Boisvert, W.F., Clark, C.W.: NIST Handbook of Mathematical Functions, NIST and Cambridge University Press, New York (2010).

22. Ryoo, C.S., A note on q-Bernoulli numbers and polynomials, Appl. Math. Lett., 20 (2007), 524-531.

23. Srivastava, H. M., Garg, M., Choudhary, S.: A new Generalization of the Bernoulli and Related Polynomials, Russ. J. Math. Phys., 17(2) (2010), 251–261.

24. Srivastava, H.M., Kim, T., Simsek, Y., q-Bernoulli numbers and polynomials associated with multiple q-Zeta functions and basic L-series, Russian J. Math. Phys., 12 (2005), 201-228.

25. Tremblay, R, Gaboury, S, Fugère, BJ: A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem. Appl. Math. Lett., 24 (2011), 1888-1893.

26. Tsumura, H., A note on q-analogues of the Dirichlet series and q-Bernoulli numbers, Jour. Num. Theory, 39(1991), 251-256.