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Desigualdades integrales fraccionales y sus *q*-análogos

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Dedicated to Professor S.L. Kalla

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Resumen

El objeto de este trabajo es establecer algunas desigualdades que envuelven operadores integrales de Saigo. Se usa el calculo q-fraccional para obtener varios resultados en la teoría de las desigualdades q-integrales. Los resultados dados anteriormente por Purohit y Raina (2013) y Sulaiman (2011) son casos especiales de los obtenidos en este trabajo.

Palabras clave: Desigualdades integrales, operadores integrales fraccionales, operadores q-integrales fraccionales.

On fractional integral inequalities and their *q*-analogues

Abstract

The aim of this paper is to establish some integral inequalities involving Saigo fractional integral operators. We then use fractional q-calculus for yielding various results in the theory of q-integral inequalities. The results given earlier by Purohit and Raina (2013) and Sulaiman (2011) follow as special cases of our findings.

Key words: Integral inequalities, fractional integral operators, fractional q-integral operators.

Introduction

Fractional integral inequalities have many applications, the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems, and in fractional partial differential equations. Further, they also provide upper and lower bounds to the solutions of the above equations. For detailed applications, one may refer to the book [1], and the recent papers [2]-[5] on the subject.

In a recent paper, Purohit and Raina [6] investigated certain Chebyshev type ([7]) integral inequalities involving the Saigo fractional integral operators, and also established the q-extensions of the main results. The aim of this paper is to establish several new integral inequalities for synchronous functions that are related to the Chebyshev functional using the Saigo fractional integral. q-Extensions of the main results are also established. Some of the results due to Purohit and Raina [6] and Sulaiman [8] follows as special cases of our results.

Following definitions will be needed in the sequel.

Definition 1. Two functions f and g are said to be synchronous on $[a,b]$, if

$$\left\{\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\right\}\geq 0, \tag{1}$$

for any $x, y \in [a,b]$.

Definition 2. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space $(C_\mu$ $(\mu \in R)$, if there exists a real number $p > \mu$ such that $f(t) = t^p \phi(t)$; where $\phi(t) \in C(0, \infty)$.

Definition 3. Let $\alpha > 0$, $\beta,\eta \in R$, then the Saigo fractional integral $I^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by ([9] see also [10, p. 19], [11]):

$$I_{0\,t}^{\alpha,\beta,\eta}\{f(t)\}=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\;{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)f(\tau)d\tau, \tag{2}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (2) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \tag{3}$$

and (t) $(a)_n$ is the Pochhammer symbol

$$(a)n = a(a+1)\cdots(a+n-1),\ (a)_0 = 1.$$

The integral operator (2) includes both the Riemann-Liouville and the Erde'lyi-Kober fractional integral operators given by the following relationships:

$$R^{\alpha}\{f(t)\}=I_{0\,t}^{\alpha,-\alpha,\eta}\{f(t)\}=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau \quad (\alpha>0) \tag{4}$$

and

$$I^{\alpha,\eta}\{f(t)\}=I_{0\,t}^{\alpha,0,\eta}\{f(t)\}=\frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\eta}f(\tau)d\tau \quad (\alpha>0,\eta\in R). \tag{5}$$

For $f(t) = t^{\mu}$ in (2), we get the known result [9]:

$$I_{0,t}^{\alpha,\beta,\eta}\left\{t^{\mu}\right\} = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{6}$$

$$(\alpha > 0, min(\mu, \mu-\beta+\eta) > -1, t > 0)$$

which shall be used in the sequel.

Fractional Integral Inequalities

The following theorems involving Saigo integral inequalities for the synchronous functions will be established.

Theorem 1. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$, $h > 0$, *then for all* $t > 0$, $\alpha > \max\{0,-\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0)$.

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}} I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\} +$$

$$I_{0,t}^{\alpha,\beta,\eta}\{g(t)\} I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\} - I_{0,t}^{\alpha,\beta,\eta}\{h(t)\} I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}. \tag{7}$$

Proof: Using Definition 1 and $h > 0$, for all $\tau, \rho \geq 0$, we have

$$\{(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho))\} \geq 0 \tag{8}$$

which implies that

$$f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) \geq f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + g(\tau)f(\rho)h(\rho)$$

$$+ g(\rho)f(\tau)h(\tau) - h(\tau)f(\rho)g(\rho) - h(\rho)f(\tau)g(\tau) \tag{9}$$

Consider

$$F(t,\tau) = \frac{t^{-\alpha-\beta}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) \quad (\tau \in (0,t); t > 0) \tag{10}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)} \frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}} +$$

$$\frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)} \frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}} + \cdots.$$

Since each term of the above series is positive in view of the conditions stated with Theorem 1, we observe that the function $F(t,\tau)$ remains positive, for all $\tau \in (0,t)$ $(t > 0)$.

Multiplying both sides of (9) by $F(t,\tau)$ (defined above by (10)) and integrating with respect to τ from 0 to t, and using (2), we get

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\}+f(\rho)g(\rho)h(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{1\}\geq g(\rho)h(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}+$$

$$f(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\}+f(\rho)h(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}+g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\}-$$

$$f(\rho)g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{h(t)\}-h(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}. \tag{11}$$

Next, multiplying both sides of (11) by $F(t,\rho)$ $(\rho \in (0,t), (t > 0)$, where $F(t,\rho)$ is given by (10), and integrating with respect to ρ from 0 to t, and using formula (6), we arrive at the desired result (7).

Theorem 2. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$*, and* $h > 0$*, then*

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\}+\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\,t^{\delta}}\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\,h(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{g(t)\,h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\,h(t)\}+$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\,h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\,g(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{f(t)\,g(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\},$$
(12)

for all $t > 0,\ \alpha > \max\{0,-\beta\},\ \gamma > \max\{0,-\delta\},\ \beta,\delta < 1,\ \beta - 1 < \eta < 0,\ \delta - 1 < \zeta < 0.$

Proof: To prove the above theorem, we start with the inequality (11). On multiplying both sides of (11) by

$$\frac{t^{-\gamma-\delta}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)}\;{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)\quad(\rho\in(0,t);\,t>0),$$

and taking integration with respect to ρ from 0 to t, we get

$$I_{0,t}^{\gamma,\delta,\zeta}\{1\}\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)\,g(t)\,h(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{1\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\,g(t)\,h(t)\}\geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\,h(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{g(t)\,h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\,h(t)\}+$$

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\,h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{h(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\,g(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{f(t)\,g(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\},$$

which on using (6) readily yields the desired result (12).

Remark 1. It may be noted that the inequalities (7) and (12) are reversed if the functions are asynchronous on $[0,\infty)$ i.e.

$$\{(f(x)-f(y))(g(x)-g(y))\}\leq 0, \tag{13}$$

for any $x, y \in [0,\infty)$.

Remark 2. For $\gamma=\alpha,\ \delta=\beta, \zeta=\eta$, Theorem 2 immediately reduces to Theorem 1.

Theorem 3. *Let* f, g *and* h *be three monotonic functions on* $[0,\infty)$ *satisfying the inequality*

$$\{(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))\}\geq 0, \tag{14}$$

then for all , $t>0, \alpha>\max\{0,-\beta\}, \gamma>\max\{0,-\delta\}, \beta,\delta<1, \beta-1<\eta<0, \delta-1<\zeta<0.$

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}} I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} \ \frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\ t^{\delta}} I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_{0,t}^{\alpha,\beta,\eta}\{g(t)h(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}I_{0,t}^{\gamma,\delta\text{a},\zeta}\{g(t)h(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{f(t)h(t)\}\ I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}-$$

$$I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)h(t)\}-I_{0,t}^{\alpha,\beta,\eta}\{h(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}I_{0,t}^{\gamma,\delta,\zeta}\{h(t)\}.$$
(15)

Proof: By applying the similar procedure as of Theorem 1 and 2, one can easily establish the above theorem. Therefore, we omit the details of the proof of this theorem.

Observe that, if we set $\beta=0$ (and $\delta=0$ additionally for Theorem 2), and make use of the relation (5), Theorems 1 to 3 respectively yield the following integral inequalities involving the Erde'lyi-Kober type fractional integral operator defined by (5):

Corollary 1. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$, *and* $h>0$, *then*

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq I^{\alpha,\eta}\{f(t)\}I^{\alpha,\eta}\{g(t)h(t)\}+I^{\alpha,\eta}\{g(t)\}I^{\alpha,\eta}\{f(t)h(t)\}$$

$$-I^{\alpha,\eta}\{h(t)\}I^{\alpha,\eta}\{f(t)g(t)\}, \tag{16}$$

for all $t>0, \alpha>0, -1<\eta<0$.

Corollary 2. *Let* f *and* g *be two synchronous functions on* $[0,\infty)$, *and* $h>0$, *then for all* $t>0, \alpha,\gamma>0, -1<\max(\eta,\zeta)<0$,

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I^{\gamma,\zeta}\{f(t)g(t)h(t)\}+\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}\ I^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I^{\alpha,\eta}\{f(t)\}I^{\gamma,\zeta}\{g(t)h(t)\}+I^{\alpha,\eta}\{g(t)h(t)\}I^{\gamma,\zeta}\{f(t)\}+I^{\alpha,\eta}\{g(t)\}I^{\gamma,\zeta}\{f(t)h(t)\}+$$

$$I^{\alpha,\eta}\{f(t)h(t)\}I^{\gamma,\zeta}\{g(t)\}-I^{\alpha,\eta}\{h(t)\}I^{\gamma,\zeta}\{f(t)g(t)\}-I^{\alpha,\eta}\{f(t)g(t)\}I^{\gamma,\zeta}\{h(t)\}. \quad (17)$$

Corollary 3. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality* (14) *then for all* $t>0,\ \alpha,\gamma>0, -1<\max(\eta,\zeta)<0$,

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}I^{\gamma,\zeta}\{f(t)g(t)h(t)\}-\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}I^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I^{\alpha,\eta}\{g(t)h(t)\}I^{\gamma,\zeta}\{f(t)\}-I^{\alpha,\eta}\{f(t)\}I^{\gamma,\zeta}\{g(t)h(t)\}+I^{\alpha,\eta}\{f(t)h(t)\}I^{\gamma,\zeta}\{g(t)\}$$

$$-I^{\alpha,\eta}\{g(t)\}I^{\gamma,\zeta}\{f(t)h(t)\}-I^{\alpha,\eta}\{h(t)\}I^{\gamma,\zeta}\{f(t)g(t)\}+I^{\alpha,\eta}\{f(t)g(t)\}I^{\gamma,\zeta}\{h(t)\}. \quad (18)$$

Again, if we replace β by $-\alpha$ and δ by $-\gamma$ in Theorems 2 and 3, and make use of the relation (4), we obtain known results due to Sulaiman [8, pp. 24-25, Theorems 2.1 to 2.2].

q-Extensions of Main Results

In this section, we establish *q*-extensions of the results derived in the previous section. We begin with the mathematical preliminaries of *q*-series and *q*-calculus. For more details of *q*-calculus and fractional *q*-calculus one can refer to [12] and [13].

The *q*-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha;q)_n=\begin{cases}1 & ; \quad n=0\\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & ; \quad n\in\mathbb{N},\end{cases} \quad (19)$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha;q)_n=\frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n>0), \quad (20)$$

where the *q*-gamma function is defined by ([12, p. 16, eqn. (1.10.1)])

$$\Gamma_q(t)=\frac{(q;q)_\infty(1-q)^{1-t}}{(q^t;q)_\infty} \quad (0<q<1). \quad (21)$$

We note that

$$\Gamma_q(1+t)=\frac{(1-q^t)\Gamma_q(t)}{1-q}, \quad (22)$$

and if $|q|<1$, the definition (19) remains meaningful for $n=\infty$, as a convergent infinite product given by

$$(\alpha;q)_\infty=\prod_{j=0}^{\infty}(1-\alpha q^j). \quad (23)$$

Also, the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu (y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{24}$$

Let $t_0 \in \mathrm{R}$, then we define a specific time scale (see [14] and [15])

$$\mathrm{T}_{t_0} = \{t; t = t_0 q^n, n \text{ a non-negative integer}\} \cup \{0\}, \quad 0 < q < 1 \tag{25}$$

and for sake of convenience, we denote T_{t_0} by T throughout this paper.

The q-derivative and q-integral of a function f defined on T are, respectively, given by (see [12, pp. 19, 22])

$$D_{q,t} f(t) = \frac{f(t) - f(tq)}{t(1-q)} \qquad (t \neq 0, q \neq 1) \tag{26}$$

and

$$\int_0^t f(\tau) d_q\tau = t(1-q)\sum_{k=0}^{\infty} q^k f(tq^k). \tag{27}$$

Definition 4. The Riemann-Liouville fractional q-integral operator of a function $f(t)$ of order α (due to [15], see also [13]) is given by

$$I_q^\alpha\{f(t)\} = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} f(\tau)\, d_q\tau \qquad (\alpha > 0, 0 < q < 1), \tag{28}$$

where

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha;q)_\infty} \quad (\alpha \in \mathrm{R}) \tag{29}$$

Definition 5. For $\alpha > 0, \eta \in \mathrm{R}$ and $0 < q < 1$, the basic analogue of the Kober fractional integral operator (cf. [16], [13]) is given by

$$I_q^{\alpha,\eta}\{f(t)\} = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1}\tau^\eta f(\tau)\, d_q\tau. \tag{30}$$

Definition 6. For $\alpha > 0$ and $\beta, \eta \in \mathrm{R}$, a basic analogue of the Saigo's fractional integral operator ([17, p. 172, eqn. (2.1)]) is given for $|\tau/t| < 1$ by

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1}$$

$$\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} q^{(\eta-\beta)m} (-1)^m \, q^{-m(m-1)/2} \left(\frac{\tau}{t}-1\right)_m f(\tau)\, d_q\tau, \tag{31}$$

which in view of (27), can be written as (see [17, p. 173, eqn. (2.5)]):

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = t^{-\beta}(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} q^{(\eta-\beta+1)m}$$
$$\times \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+m};q)_k}{(q;q)_k} \; f(tq^{k+m}). \tag{32}$$

In the sequel, we shall be using the following image formula ([17, p. 173, eqn. (2.11)]):

$$I_q^{\alpha,\beta,\eta}\{t^{\mu}\} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu+1-\beta+\eta)}{\Gamma_q(\mu+1-\beta)\Gamma_q(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{33}$$

$$(\alpha > 0, 0 < q < 1, \min(\mu, \mu-\beta+\eta) > -1, t > 0).$$

Now, we shall establish new q-integral inequalities for the synchronous functions involving the fractional q-integral operators, which can be treated as the q-analogues of the inequalities (7), (12) and (15).

Theorem 4. *Let f and g be two synchronous functions, and $h(t) > 0$ on T, then*

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\, t^{\beta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\} I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}, \tag{34}$$

where $t > 0, 0 < q < 1, \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0)$.

Proof: By the hypothesis, the functions f and g are synchronous functions on T for all $\tau, \rho \geq 0$, and $h(t) > 0$ therefore the inequality (9) is satisfied, that is

$$f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) \geq f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + g(\tau)f(\rho)h(\rho)$$
$$+ g(\rho)f(\tau)h(\tau) - h(\tau)f(\rho)g(\rho) - h(\rho)f(\tau)g(\tau)$$

Since, $\tau \in (0,t)$ $(t > 0)$, $0 < q < 1$, then $tq^{k+m} \in (0, t)$ for $k, m \in N$, therefore, on replacing τ by tq^{k+m} *in the above inequality we get*

$$f(tq^{k+m})g(tq^{k+m})h(tq^{k+m}) + f(\rho)\, g(\rho)\, h(\rho) \geq f(tq^{k+m})\, g(\rho)\, h(\rho) +$$

$$f(\rho)g(tq^{k+m})h(tq^{k+m}) + g(tq^{k+m})f(\rho)h(\rho) + g(\rho)f(tq^{k+m})h(tq^{k+m}) -$$

$$h(tq^{k+m})f(\rho)g(\rho) - h(\rho)f(tq^{k+m})g(tq^{k+m}). \tag{35}$$

Consider

$$\mathsf{H}(t,q) = t^{-\beta}(1-q)^{\alpha}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{(q^{\alpha+m};q)_k}{(q;q)_k}q^{(\eta-\beta+1)m+k}\,(k,m\in N). \tag{36}$$

Evidently, under the conditions stated with Theorem 4, we observe that the function $\mathsf{H}(t,q)$ is positive for all values of $k,m\in N$. Therefore, on multiplying both sides of (35) by $\mathsf{H}(t,q)$ and taking summations between the limits $k = 0$ *to* ∞, we get

$$\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})g(tq^{k+m})h(tq^{k+m}) + f(\rho)g(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q) \geq$$

$$g(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m}) + f(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)g(tq^{k+m})h(tq^{k+m}) +$$

$$f(\rho)h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)g(tq^{k+m}) + g(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})h(tq^{k+m}) -$$

$$f(\rho)g(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)h(tq^{k+m}) - h(\rho)\sum_{k=0}^{\infty}\mathsf{H}(t,q)f(tq^{k+m})g(tq^{k+m}).$$

Now, on again taking summation from $m = 0$ *to* ∞, and then making use of the definition (32), we obtain

$$I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} + f(\rho)g(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{1\} \geq g(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)\} +$$

$$f(\rho)\,I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} + f(\rho)h(\rho)\,I_q^{\alpha,\beta,\eta}\{g(t)\} + g(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} -$$

$$f(\rho)g(\rho)\,I_q^{\alpha,\beta,\eta}\{h(t)\} - h(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}. \tag{37}$$

Next, in the above inequality on replacing ρ by tq^{k+m}, multiplying both sides of by $\mathsf{H}(t,q)$, taking summations between the limits $k = 0$ *to* ∞, and $m = 0$ *to* ∞, and then making use of the definitions (32) and (33), we arrive at the desired inequality (34).

Theorem 5. *Let* f *and* g *be two synchronous functions on* T*, and* $h > 0$*, then for all* $t > 0, 0 < q < 1$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta, \delta < 1$, $\beta - 1 < \eta < 0$, $\delta - 1 < \xi < 0$,

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} + \frac{\Gamma_q(1-\delta+\zeta)}{\Gamma_q(1-\delta)\Gamma_q(1+\gamma+\zeta)\,t^{\delta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_q^{\alpha,\beta,\eta}\{f(t)g(t)\} I_q^{\gamma,\delta,\zeta}\{h(t)\}. \qquad (38)$$

Proof: To prove the above theorem, we start with the inequality (37). On replacing ρ by tq^{k+m} and multiplying both sides by a positive function $\mathsf{F}(t,q)$, given by

$$\mathsf{F}(t,q) = t^{-\delta}(1-q)^{\gamma} \frac{(q^{\gamma+\delta};q)_m (q^{-\zeta};q)_m}{(q^{\gamma};q)_m (q;q)_m} \frac{(q^{\gamma+m};q)_k}{(q;q)_k} q^{(\zeta-\delta+1)m+k} \, (k, m \in N). \qquad (39)$$

taking summations between the limits $k = 0$ *to* ∞ and $m = 0$ *to* ∞, and then making use of the definition (32), then the inequality (37) leads to

$$I_q^{\gamma,\delta,\zeta}\{1\} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{1\} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} \geq$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)\} + I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} +$$

$$I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{h(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\} - I_q^{\alpha,\beta,\eta}\{f(t)g(t)\} I_q^{\gamma,\delta,\zeta}\{h(t)\}, \qquad (40)$$

which yields the desired result by taking (33) into account.

Remark 3. The inequalities (34) and (38) are reversed if the functions are asynchronous on T.

Remark 4. Again, when $\gamma = \alpha$, $\delta = \beta$, $\zeta = \eta$, then Theorem 5 leads to Theorem 4.

Theorem 6. *Let* f *,* g *and* h *be three monotonic functions on* $[0,\infty)$ *satisfying the inequality* (14), *then for all* $t > 0, 0 < q < 1$, $\alpha > \max\{0,-\beta\}$, $\gamma > \max\{0,-\delta\}$, $\beta,\delta < 1$, $\beta-1 < \eta < 0, \delta-1 < \zeta < 0$, *we have*

$$\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}} I_q^{\gamma,\delta,\zeta}\{f(t)g(t)h(t)\} -$$

$$\frac{\Gamma_q(1-\delta+\zeta)}{\Gamma_q(1-\delta)\Gamma_q(1+\gamma+\zeta)\,t^{\beta}} I_q^{\alpha,\beta,\eta}\{f(t)g(t)h(t)\} \geq I_q^{\alpha,\beta,\eta}\{g(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)\} -$$

$$I_q^{\alpha,\beta,\eta}\{f(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)h(t)\} + I_q^{\alpha,\beta,\eta}\{f(t)h(t)\} I_q^{\gamma,\delta,\zeta}\{g(t)\} - I_q^{\alpha,\beta,\eta}\{g(t)\} I_q^{\gamma,\delta,\zeta}\{f(t)h(t)\} -$$

$$I_q^{\alpha,\beta,\eta}\{h(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}+I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}I_q^{\gamma,\delta,\zeta}\{h(t)\}. \tag{41}$$

Proof: By applying the same procedure as of Theorem 4 and 5, one can establish the above theorem. Therefore, we omit the details of the proof.

Now, if we set $\beta = 0$ (and additionally $\delta = 0$ for Theorem 5), and make use of the known result [18, p.173, eqn. (2.9)], namely

$$I_q^{\alpha,0,\eta}\{f(t)\}=I_q^{\alpha,\eta}\{f(t)\}, \tag{42}$$

Theorems 4 to 6 respectively reduce to the following q-integral inequalities involving the Erde'lyi-Kober type fractional q-integral operators:

Corollary 4. *Let* f *and* g *be two synchronous functions on* T, *and* $h>0$, *then*

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq I_q^{\alpha,\eta}\{f(t)\}I_q^{\alpha,\eta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{g(t)\}I_q^{\alpha,\eta}\{f(t)h(t)\}$$

$$-I_q^{\alpha,\eta}\{h(t)\}I_q^{\alpha,\eta}\{f(t)g(t)\}, \tag{43}$$

for all $t>0, 0<q<1, \alpha>0$ and $-1<\eta<0$).

Corollary 5. *Let* f *and* g *be two synchronous functions on* T, *and* $h>0$, *then for* $t>0$, $0<q<1$, $\alpha, \gamma>0$, *such that* $-1<\max(\eta,\zeta)<0$),

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\gamma,\zeta}\{f(t)g(t)h(t)\}+\frac{\Gamma_q(1+\zeta)}{\Gamma_q(1+\gamma+\zeta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_q^{\alpha,\eta}\{f(t)\}I_q^{\gamma,\zeta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{g(t)h(t)\}I_q^{\gamma,\zeta}\{f(t)\}+I_q^{\alpha,\eta}\{g(t)\}I_q^{\gamma,\zeta}\{f(t)h(t)\}+$$

$$I_q^{\alpha,\eta}\{f(t)h(t)\}I_q^{\gamma,\zeta}\{g(t)\}-I_q^{\alpha,\eta}\{h(t)\}I_q^{\gamma,\zeta}\{f(t)g(t)\}-I_q^{\alpha,\eta}\{f(t)g(t)\}I_q^{\gamma,\zeta}\{h(t)\}. \tag{44}$$

Corollary 6. *Let* f, g *and* h *be three monotonic functions on* $[0,\infty)$ *satisfying the inequality* (14), *then for all* $t>0$, $\alpha, \gamma, >0, -1<\max(\eta,\zeta)<0, 0<q<1$

$$\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}I_q^{\gamma,\zeta}\{f(t)g(t)h(t)\}-\frac{\Gamma_q(1+\zeta)}{\Gamma_q(1+\gamma+\zeta)}I_q^{\alpha,\eta}\{f(t)g(t)h(t)\}\geq$$

$$I_q^{\alpha,\eta}\{g(t)h(t)\}I_q^{\gamma,\zeta}\{f(t)\}-I_q^{\alpha,\eta}\{f(t)\}I_q^{\gamma,\zeta}\{g(t)h(t)\}+I_q^{\alpha,\eta}\{f(t)h(t)\}I_q^{\gamma,\zeta}\{g(t)\}$$

$$-I_q^{\alpha,\eta}\{g(t)\}I_q^{\gamma,\zeta}\{f(t)h(t)\}-I_q^{\alpha,\eta}\{h(t)\}I_q^{\gamma,\zeta}\{f(t)g(t)\}+I_q^{\alpha,\eta}\{f(t)g(t)\}I_q^{\gamma,\zeta}\{h(t)\}. \tag{45}$$

Further, we observe that, if we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation [17, p.173, eqn. (2.7)], namely

$$I_q^{\alpha,-\alpha,\eta}\{f(t)\} = I_q^{\alpha}\{f(t)\} \tag{46}$$

and

$$I_q^{\gamma,-\gamma,\zeta}\{f(t)\} = I_q^{\gamma}\{f(t)\}, \tag{47}$$

then, Theorems 4 to 6 reduce to the following q-integral inequalities involving the Riemann-Liouville type of fractional q-integral operators.

Corollary 7. *Let f and g be two synchronous functions on* T, *and $h > 0$, then*

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{f(t)\} I_q^{\alpha}\{g(t)h(t)\} + I_q^{\alpha}\{g(t)\} I_q^{\alpha}\{f(t)h(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\alpha}\{f(t)g(t)\}, \tag{48}$$

for all $t > 0, 0 < q < 1$ and $\alpha > 0$.

Corollary 8. *Let f and g be two synchronous functions on* T, *and $h > 0$, then for* $t > 0,\ 0 < q < 1,\ \alpha, \gamma > 0$,

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\gamma}\{f(t)g(t)h(t)\} + \frac{t^{\gamma}}{\Gamma_q(1+\gamma)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)h(t)\} + I_q^{\alpha}\{g(t)h(t)\} I_q^{\gamma}\{f(t)\} + I_q^{\alpha}\{g(t)\} I_q^{\gamma}\{f(t)h(t)\} + I_q^{\alpha}\{f(t)h(t)\} I_q^{\gamma}\{g(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\gamma}\{f(t)g(t)\} - I_q^{\alpha}\{f(t)g(t)\} I_q^{\gamma}\{h(t)\}. \tag{49}$$

Corollary 9. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality* (14), *then for all* $t > 0,\ \alpha, \gamma > 0, 0 < q < 1$,

$$\frac{t^{\alpha}}{\Gamma_q(1+\alpha)} I_q^{\gamma}\{f(t)g(t)h(t)\} - \frac{t^{\gamma}}{\Gamma_q(1+\gamma)} I_q^{\alpha}\{f(t)g(t)h(t)\} \geq I_q^{\alpha}\{g(t)h(t)\} I_q^{\gamma}\{f(t)\} - I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)h(t)\} + I_q^{\alpha}\{f(t)h(t)\} I_q^{\gamma}\{g(t)\} - I_q^{\alpha}\{g(t)\} I_q^{\gamma}\{f(t)h(t)\} - I_q^{\alpha}\{h(t)\} I_q^{\gamma}\{f(t)g(t)\} + I_q^{\alpha}\{f(t)g(t)\} I_q^{\gamma}\{h(t)\}. \tag{50}$$

Special Cases

We now, briefly consider some of the consequences of the results derived in the previous sections. If we let $q \to 1^{-}$, and use the limit formulas:

$$\lim_{q \to 1^-} \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n \tag{51}$$

and

$$\lim_{q \to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha), \tag{52}$$

the results of Section 3 correspond to the results obtained in Section 2. Again, in view of the above limiting cases, Corollaries 8 and 9 provide, respectively, the q-extensions of the inequalities due to Sulaiman [8, pp. 24-25, Theorems 2.1 to 2.2].

Finally, if we consider the function h as constant > 0, the Theorems 1, 2, 4 and 5, and Corollaries 7 and 8 provide, respectively, the known results due to Purohit and Raina [6], and Öğünmez and Özkan [14].

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References

1. G.A. Anastassiou, *Advances on Fractional Inequalities*, Springer Briefs in Mathematics, Springer, New York, 2011.
2. Z. Denton, A.S. Vatsala, *Monotonic iterative technique for finit system of nonlinear Riemann-Liouville fractional differential equations*, Opuscula Mathematica, **31(3)**(2011), 327-339.
3. S.L. Kalla and Alka Rao, *On Grüss type inequality for hypergeometric fractional integrals*, Le Matematiche, **66 (1)**(2011), 57-64.
4. V. Lakshmikantham and A.S. Vatsala, *Theory of fractional differential inequalities and applications*, Commu. Appl. Anal., **11** (2007), 395-402.
5. J.D. Ramírez, A.S. Vatsala, *Monotonic iterative technique for fractional differential equations with periodic boundary conditions*, Opuscula Mathematica, **29(3)**(2009), 289-304.
6. S.D. Purohit and R.K. Raina, *Chebyshev type inequalities for the Saigo fractional integrals and their q-analogues*, J. Math. Inequal., **7(2)** (2013), 239-249.
7. P.L. Chebyshev, *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov, **2**(1882), 93-98.
8. W.T. Sulaiman, *Some new fractional integral inequalities*, J. Math. Anal., **2(2)** (2011), 23-28.
9. M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., **11** (1978) 135-143.
10. V.S. Kiryakova, *Generalized Fractional Calculus and Applications* (Pitman Res. Notes Math. Ser. 301), Longman Scientific & Technical, Harlow, 1994.
11. R.K. Raina, *Solution of Abel-type integral equation involving the Appell hypergeometric function*, Integral Transforms Spec. Funct., **21(7)**(2010), 515-522.
12. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

13. M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations* (Lecture Notes in Mathematics 2056), Springer-Verlag Berlin Heidelberg, 2012.

14. H. Öğünmez, and U.M. Özkan, *Fractional quantum integral inequalities*, J. Inequal. Appl., Volume 2011, Article ID 787939, 7 pp.

15. R.P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Camb. Phil. Soc., **66** (1969), 365-370.

16. W.A. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edin. Math. Soc., **15**(1966), 135-140.

17. M. Garg and Lata Chanchlani, *q-Analogue of Saigo's fractional calculus operators*, Bull. Math. Anal. Appl., **3(4)** (2011), 169-179.