Revista Tecnocientífica URU
Universidad Rafael Urdaneta
Facultad de Ingeniería
Nº 6 Enero - Junio 2014
Depósito legal: ppi 201402ZU4464
ISSN: 2343 - 6360

Algunos resultados que involucran operadores q-integrales fraccionales generalizados de Erdélyi-Kober

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(Dedicated to Professor Shyam Lal Kalla in occasion of his 76 years)

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Recibido: 28-10-2013 Aceptado: 04-02-2014

Resumen

En este trabajo se presentan algunos resultados para los operadores q-integrales fraccionales generalizados de Erdélyi-Kober definidos por Galué (2009). Además, se establecen desigualdades q-integrales fraccionales para funciones sincrónicas usando los operadores q-integrales fraccionales generalizados de Erdélyi-Kober antes mencionados. Algunos resultados dados por Belarbi y Dahmani, Öğünmez y Özkan, y Sulaiman se derivan como casos especiales de nuestros resultados.

Palabras clave: Operador q-integral fraccional generalizado de Erdélyi-Kober, desigualdades q-integrales fraccionales, q-integración fraccional por partes.

Some results involving generalized Erdélyi-Kober fractional q-integral operators

Abstract

In this paper some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué (2009) are presented. Also, fractional q-integral inequalities for synchronous functions are established using generalized Erdélyi-Kober fractional q-integral operators earlier mentioned. Some results due to Belarbi and Dahmani, Öğünmez and Özkan, and Sulaiman follow as special cases of our results.

Key words: Generalized Erdélyi-Kober fractional q-integral operator, fractional q-integral inequalities, fractional q-integration by parts.

Introduction

The most widely used definition of an integral of fractional order is via an integral transform, called the Riemann-Liouville operator of fractional integration: [1, p. 146]

$$\begin{aligned} {}_aI_x^{\alpha}\varphi(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt, \quad \mathrm{Re}(\alpha)>0, \\ &= \frac{d^n}{dx^n}\, {}_aI_x^{\alpha+n}\varphi(x), \quad -n<\mathrm{Re}(\alpha)\le 0,\ n\in \mathbb{N}. \end{aligned} \tag{1}$$

Many authors, including Agarwal [2], Al-Salam [3], Kalia [4], Galué ([5]-[7]), Kalla *et al.* [8], Kalla and Kiryakova [9], Kiryakova [10], McBride and Roach [11], Ross [1], Saigo [12], Samko *et al.* [13], Saxena *et al.* [14], have defined and studied operators of fractional integration with their applications. We mention here some of these operators:

Erdélyi-Kober Operator: [3, p. 4, Eq. (20)]

$$\begin{aligned} I_{\eta,\alpha}f(x) &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt, \ \mathrm{Re}(\alpha)>0, \\ &= x^{-\alpha-\eta}\frac{d^n}{dx^n}\, x^{\eta+\alpha+n} I_{\eta,\alpha+n}f(x), \ -n<\mathrm{Re}(\alpha)\le 0. \end{aligned} \tag{2}$$

Basic analogue of Riemann-Liouville integral operator

Introduced by Al-Salam through [3]

$$I_q^{\alpha}f(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x (x-tq)_{\alpha-1}\, f(t)\,d_qt, \quad \alpha\in \mathrm{C},\ \mathrm{Re}(\alpha)>0. \tag{3}$$

q-analogue of Liouville fractional integral operator

The fractional q-integral operator $K_q^{-\alpha}$ is a q-analogue of Liouville fractional integral and it is defined by [3]

$$K_q^{-\alpha}f(x)=\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}\int_x^{\infty}(t-x)_{\alpha-1}\, f(q^{1-\alpha}t)\,d_qt, \quad \alpha\in \mathrm{C},\ \mathrm{Re}(\alpha)>0. \tag{4}$$

Basic analogue of Kober fractional integral operator

A basic analogue of Kober fractional integral operator has been defined by Agarwal [2] in the following form:

$$I_q^{\eta,\mu}f(x)=\frac{x^{-\eta-\mu}}{\Gamma_q(\mu)}\int_0^x t^{\eta}(x-tq)_{\mu-1}\, f(t)\,d_qt, \quad \eta,\mu\in \mathrm{C},\ \mathrm{Re}(\mu)>0, \tag{5}$$

where the order of integration μ is arbitrary real or complex number, and

$$(x-y)_{\nu}=x^{\nu}\prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^{n}}{1-(y/x)q^{n+\nu}}\right]. \tag{6}$$

The result (5) can be expressed as [14]

$$I_q^{\eta,\mu} f(x) = \frac{(1-q)}{\Gamma_q(\mu)} \sum_{k=0}^{\infty} q^{k(1+\eta)} \left(1-q^{k+1}\right)_{\mu-1} f(xd^k). \tag{7}$$

Basic analogue of Weyl fractional integral operator

A basic analogue of Weyl fractional integral operator has been defined by Al-Salam [3] as follows:

$$K_q^{\eta,\mu} f(x) = \frac{q^{-\eta} x^{\eta}}{\Gamma_q(\mu)} \int_x^{\infty} (t-x)_{\mu-1} t^{-(\eta+\mu)} f(q^{(1-\mu)} t) d_q t, \quad \eta \in \mathrm{C},\ \mathrm{Re}(\mu) > 0. \tag{8}$$

The generalized Erdélyi-Kober fractional q-integral operator: Defined by [5]

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^x \left(x^{\beta} - t^{\beta} q\right)_{\mu-1} t^{\beta(\eta+1)-1} f(t)\, d_q t \tag{9}$$

$$= \beta(1-q^{1/\beta})(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^{\mu};q)_k}{(q;q)_k} q^{k(\eta+1)} f(xq^{k/\beta}). \tag{10}$$

$\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ \eta \in \mathrm{C}.$

As particular case of (9) we have

$$I_q^{0,\mu,1} f(x) = I_q^{0,\mu} f(x) = x^{-\mu} I_q^{\mu} f(x). \tag{11}$$

The generalized Weyl fractional q-integral operator: [6]

$$K_q^{\eta,\mu,\beta} f(x) = \frac{\beta q^{-\eta} x^{\beta\eta}}{\Gamma_q(\mu)} \int_x^{\infty} (t^{\beta} - x^{\beta})_{\mu-1} t^{-\beta(\eta+\mu-1)-1} f(q^{(1-\mu)/\beta} t) d_q t \tag{12}$$

$$= \beta(1-q^{1/\beta})(1-q)^{\mu-1} \sum_{k=0}^{\infty} \frac{(q^{\mu};q)_k}{(q;q)_k} q^{kn} f(xq^{-(\mu+k)/\beta}) \tag{13}$$

$$\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ \eta \in \mathrm{C}.$$

On the other hand, various researchers in the field of integral inequalities, motived by the usefulness of the fractional integral inequalities in fractional partial differential equations and in the solutions of fractional boundary value problems ([15]-[18]), have explored certain extensions and generalizations by involving fractional calculus operators. See for example references [15], [19]-[27].

In this paper some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué [5] are presented. Also, fractional q-integral inequalities for synchronous functions are established using generalized Erdélyi-Kober fractional q-integral operators earlier mentioned. Some results due to Belarbi and Dahmani [19], Öğünmez and Özkan [24], and Sulaiman [27] follow as special cases of our results.

Preliminares

In this section, we present some basic definitions, useful in our analysis.

The q-analogue of a complex number a is defined by [28]

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \in \mathbb{C} \setminus \{1\} \tag{14}$$

The q-shifted factorial is defined as [29]

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), & n = 1, 2\dots \\ [(1-aq^{-1})(1-aq^{-2})\dots(1-aq^{-n})]^{-1}, & n = -1,-2\dots \end{cases} \tag{15}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k), \tag{16}$$

which converges for $|q| < 1$ and diverges for $a \neq 0$ and $|q| ?1,$ and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in Z, |q| < 1. \tag{17}$$

q-Gamma function

It is defined as follows: [29, p. 235, Eq. (I.35)]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1. \tag{18}$$

Obviously,

$$\Gamma_q(x+1) = [x]_q \, \Gamma_q(x)$$

Basic hypergeometric series

This series is due to Heine (1846), [29]

$$_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \tag{19}$$

where it is assumed that $c \neq q^{-m}$ for $m = 0,1,\dots,$ and $(a;q)_n$ is the q-shifted factorial defined in (15).

The q-binomial theorem

One of the important summation formulae for hypergeometric series is given by the following binomial theorem:

$$_2F_1(a,c;c;z) = {}_1F_0(a;-;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}, \quad |z| < 1,$$

whose q-analogue was derived by Cauchy (1843), Heine (1847) and others [29]

$$ {}_1\phi_0(a;-;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;\, q)_\infty}, \quad |z|<1, |q|<1. \tag{20}$$

The q-derivative operator

This is denoted by D_q and defined for fixed q as [29, p. 22]

$$ \mathrm{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)\, z}, \quad z \neq 0, q \neq 1, \qquad \mathrm{D}_q f(0) = \lim_{z \to 0} \mathrm{D}_q f(z). \tag{21}$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathrm{R}$, we have linearity and product rules [30]

$$ \mathrm{D}_q\big(\alpha\, u(x) + \beta\, v(x)\big) = \alpha\big(\mathrm{D}_q u\big)(x) + \beta\big(\mathrm{D}_q v\big)(x) \tag{22}$$

$$ \mathrm{D}_q\big(u(x)\cdot v(x)\big) = u(qx)\big(\mathrm{D}_q v\big)(x) + v(x)\big(\mathrm{D}_q u\big)(x). \tag{23}$$

The q-integral

Thomae (1869) and Jackson (1910) introduced the q-integral in the following form [29, p. 19, Eqs. (1.11.2), (1.11.3)]

$$ \int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, $$

where

$$ \int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n. \tag{24}$$

The q-integration by parts

The following is the formula for the q-integration by parts [31]

$$ \int_a^b f(x)\big(\mathrm{D}_q g\big)(x)\, d_q x = \big[f(x)\, g(x)\big]_a^b - \int_a^b g(qx)\big(\mathrm{D}_q f\big)(x)\, d_q x. \tag{25}$$

Further Results For Generalized Erdélyi-Kober Fractional Q-integral Operators

In this section we establish some results for generalized Erdélyi-Kober fractional q-integral operators defined by Galué [5].

Theorem 1. For $\mathrm{Re}(\beta), \mathrm{Re}(\mu) > 0,\ 0 < q < 1,$

i) If $\eta = 0,$ then

$$ I_q^{0,\mu,\beta} f(x) = \frac{\beta[1/\beta]_q}{[\mu]_q\, \Gamma_q(\mu)} f(0) + I_q^{0,\mu+1,\beta} \mathrm{D}_q f(x), \tag{26}$$

provided that $f(0)$ exists.

ii) If $\mathrm{Re}(\eta) > 0,$ then

$$I_q^{\eta,\mu,\beta} f(x) = q^{\eta}\ I_q^{\eta,\mu+1,\beta} \mathrm{D}_q\big(f(x)\big) + x^{-\beta}\ [\eta]_q I_q^{\eta-1,\mu+1,\beta} f(x) \tag{27}$$

Proof. Making in (9) a change of variable and using (21) and (14), we get

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta[1/\beta]_q\ x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)} \int_0^{x^\beta} \big(x^\beta - yq\big)_{\mu-1}\ y^{\eta}\ f(y^{1/\beta})\ d_q y. \tag{28}$$

Since that [30]

$$\mathrm{D}_{q,t}(x-t)_\alpha = -[\alpha]_q\ (x-tq)_{\alpha-1}$$

we can write

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q\ x^{-\beta(\eta+\mu)}}{[\mu]_q\ \Gamma_q(\mu)} \int_0^{x^\beta} \mathrm{D}_{q,y}\big(x^\beta - y\big)_{\mu}\ y^{\eta}\ f(y^{1/\beta})\ d_q y.$$

Now, using the q-integration by parts (25), we have

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q\ x^{-\beta(\eta+\mu)}}{[\mu]_q\ \Gamma_q(\mu)} \Big\{ \Big[\big(x^\beta - y\big)_{\mu}\ y^{\eta}\ f\big(y^{1/\beta}\big)\Big]_0^{x^\beta} -$$

$$\int_0^{x^\beta} \big(x^\beta - yq\big)_{\mu}\ \mathrm{D}_q\big(y^{\eta}\ f(y^{1/\beta})\big)\ d_q y\Big\}. \tag{29}$$

From this result making $\eta = 0,$ after of some calculations and using (28) we arrive to the result (26).

On the other hand, by applying the rule of the derivative of a product (23), (21) and (14)

$$\mathrm{D}_q\big(y^{\eta}\ f(y^{1/\beta})\big) = q^{\eta}\ y^{\eta}\ \mathrm{D}_q\big(f(y^{1/\beta})\big) + [\eta]_q\ y^{\eta-1}\ f(y^{1/\beta}) \tag{30}$$

therefore, from (29) and (30)

$$I_q^{\eta,\mu,\beta} f(x) = -\frac{\beta[1/\beta]_q\ x^{-\beta(\eta+\mu)}}{[\mu]_q\ \Gamma_q(\mu)} \Big\{ \Big[\big(x^\beta - y\big)_{\mu}\ y^{\eta}\ f\big(y^{1/\beta}\big)\Big]_0^{x^\beta} -$$

$$q^{\eta} \int_0^{x^\beta} \big(x^\beta - yq\big)_{\mu}\ y^{\eta}\ \mathrm{D}_q\big(f(y^{1/\beta})\big)\ d_q y - [\eta]_q \int_0^{x^\beta} \big(x^\beta - yq\big)_{\mu}\ y^{\eta-1}\ f(y^{1/\beta})\ d_q y\Big\}.$$

From this expression, after of some evaluations and using (28), we get the result (27).

Theorem 2. If $\mathrm{Re}(\beta) > 0, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\eta + 1 + \nu/\beta) > 0, \eta \in \mathbb{C}, 0 < q < 1$ then

$$I_q^{\eta,\mu,\beta}\{x^{\nu}\} = \beta\,[1/\beta]_q\ \frac{\Gamma_q(\eta + 1 + \nu/\beta)}{\Gamma_q(\mu + \eta + 1 + \nu/\beta)} x^{\nu}. \tag{31}$$

Proof. From (10) we get

$$I_q^{\eta,\mu,\beta}\left\{x^{\nu}\right\}=\beta(1-q^{1/\beta})(1-q)^{\mu-1}\ x^{\nu}\sum_{k=0}^{\infty}\frac{(q^{\mu};q)_k}{(q;q)_k}q^{k(\eta+1+\nu/\beta)}$$

now, using the q-binomial theorem given in (20) we have

$$I_q^{\eta,\mu,\beta}\left\{x^{\nu}\right\}=\beta(1-q^{1/\beta})(1-q)^{\mu-1}\ x^{\nu}\ {}_1\phi_0\left(q^{\mu};-;q,q^{\eta+1+\nu/\beta}\right)$$
$$=\beta(1-q^{1/\beta})(1-q)^{\mu-1}\frac{\left(q^{\mu+\eta+1+\nu/\beta};q\right)_{\infty}}{\left(q^{\eta+1+\nu/\beta};q\right)_{\infty}}x^{\nu},$$

finally, the application of (17) and (14) to this result leads us to (31).

Lemma 1. If K is a constant and $\mathrm{Re}(\beta)>0,\mathrm{Re}(\mu)>0,\ \mathrm{Re}(\eta)>-1, 0<q<1,$ then

$$I_q^{\eta,\mu,\beta}\left\{K\right\}=\beta\,[1/\beta]_q\,\frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)}K. \tag{32}$$

Proof. From (9) we obtain

$$I_q^{\eta,\mu,\beta}\left\{K\right\}=K\ I_q^{\eta,\mu,\beta}\left\{1\right\}$$

and using (31) with $\nu=0$ we obtain of desired result.

Theorem 3. If $\mathrm{Re}(\beta),\mathrm{Re}(\mu)>0,\ \eta\in\mathbb{C}, 0<q<1$ then we have the following result for fractional q-integration by parts

$$\int_0^{\infty}g(q^{-\mu/\beta}x)\ I_q^{\eta-1+1/\beta,\mu,\beta}f(x)\,d_qx=\int_0^{\infty}f(x)\ K_q^{\eta,\mu,\beta}g(x)\,d_qx. \tag{33}$$

provided that the q-integrals exist.

Proof. Let be

$$I=\int_0^{\infty}g(q^{-\mu/\beta}x)\ I_q^{\eta-1+1/\beta,\mu,\beta}f(x)\,d_qx.$$

Using the definition of the operatos $I_q^{\eta,\mu,\beta}(.)$ given in (10) and interchanging the order of integration and summation, assuming absolute convergence, we get

$$I=\beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1}\sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k}q^{k(\eta+1/\beta)}\int_0^{\infty}g(q^{-\mu/\beta}x)f\left(xq^{k/\beta}\right)d_qx,$$

now, making a simple variable change and using the result (21), we obtain

$$I=\beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1}\sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k}q^{k\eta}\int_0^{\infty}g(q^{(-\mu-k)/\beta}w)f(w)\,d_qw,$$

and interchanging the order of summation and integration we have

$$I = \int_0^\infty f(w)\left[\beta\left(1-q^{1/\beta}\right)(1-q)^{\mu-1}\sum_{k=0}^{\infty}\frac{\left(q^{\mu};q\right)_k}{\left(q;q\right)_k}q^{k\eta}g(q^{(-\mu-k)/\beta}w)\right]d_q w.$$

Finally, interpreting this expression in terms of the operator $K_q^{\eta,\mu,\beta}(.)$ we obtain (33).

Particular cases: i) From (33) with $\beta = 1$:

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\eta,\mu} f(x)\, d_q x = \int_0^\infty f(x)\, K_q^{\eta,\mu} g(x)\, d_q x. \tag{34}$$

ii) Put $\eta = -\mu$ in (34) and using (3)-(4),

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\mu}(x^{-\mu} f(x)\ \ d_q x = q^{\mu(\mu+1)/2}\int_0^\infty x^{-\mu} f(x)\, K_q^{-\mu} g(x)\, d_q x,$$

that is,

$$\int_0^\infty g(q^{-\mu}x)\, I_q^{\mu} F(x)\, d_q x = q^{\mu(\mu+1)/2}\int_0^\infty F(x)\, K_q^{-\mu} g(x)\, d_q x, \tag{35}$$

which is a known result [32, p. 159, No. (5.25)].

Fractional q-integral inequalities

In this section, we establish some fractional q-integral inequalities employing generalized Erdélyi-Kober fractional q-integral operators defined in (9).

Definition. Two functions f and g are said to be synchronous on $[a, b]$, if

$$(f(x)-f(y))(g(x)-g(y)\ \geq 0, \tag{36}$$

for any $x, y \in [a, b]$.

Theorem 4. Let p be a positive function on $[0,\infty)$, f , g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\begin{aligned}\alpha[1/\alpha]_q \frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)} I_q^{\eta,\mu,\beta}\left(f(x)g(x)p(x)\right) + \beta[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu,\alpha}\left(f(x)g(x)p(x)\right) \geq \\ I_q^{\varepsilon,\nu,\alpha}\left(g(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)p(x)\right) + I_q^{\varepsilon,\nu,\alpha}\left(g(x)p(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)\right) + \\ I_q^{\varepsilon,\nu,\alpha}\left(f(x)\right)\, I_q^{\eta,\mu,\beta}\left(g(x)p(x)\right) + I_q^{\varepsilon,\nu,\alpha}\left(f(x)p(x)\right)\, I_q^{\eta,\mu,\beta}\left(g(x)\right) - \\ I_q^{\varepsilon,\nu,\alpha}\left(p(x)\right)\, I_q^{\eta,\mu,\beta}\left(f(x)g(x)\right) - I_q^{\varepsilon,\nu,\alpha}\left(f(x)g(x)\right)\, I_q^{\eta,\mu,\beta}\left(p(x)\right)\end{aligned} \tag{37}$$

where $x > 0,\ 0 < q < 1,\ \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\nu) > 0, \mathrm{Re}(\eta) > -1, \mathrm{Re}(\varepsilon) > -1.$

Proof. Since the functions f and g are synchronous functions on $[0,\infty)$ therefore from definition (36), we have

$$\left(f(t)-f(y)\right)\left(g(t)-g(y)\right) \geq 0, \qquad t, y \in [0,\infty)$$

then as p is a positive function on $[0,\infty)$

$$\left(f(t)-f(y)\right)\left(g(t)-g(y)\right)\left(p(t)+p(y)\right)\geq 0,$$

that is,

$$f(t)g(t)p(t)+f(t)g(t)p(y)+f(y)g(y)p(t)+f(y)g(y)p(y)\geq$$

$$f(t)g(y)p(t)+f(t)g(y)p(y)+f(y)g(t)p(t)+f(y)g(t)p(y) \tag{38}$$

Let be

$$F(\eta,\mu,\beta;x,t)=\frac{\beta x^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)}\left(x^\beta-t^\beta q\right)_{\mu-1}t^{\beta(\eta+1)-1} \quad x>0, t\in(0,x) \tag{39}$$

Observe from (6) that

$$F(\eta,\mu,\beta;x,t)=\frac{\beta}{\Gamma_q(\mu)}\,x^{-\beta(\eta+1)}\,t^{\beta(\eta+1)-1} \qquad \times$$

$$\frac{\left(1-\frac{t^\beta}{x^\beta}q\right)\left(1-\frac{t^\beta}{x^\beta}q^2\right)\cdots\left(1-\frac{t^\beta}{x^\beta}q^{k+1}\right)\cdots}{\left(1-\frac{t^\beta}{x^\beta}q^\mu\right)\left(1-\frac{t^\beta}{x^\beta}q^{\mu+1}\right)\cdots\left(1-\frac{t^\beta}{x^\beta}q^{\mu+k}\right)\cdots}, \quad k=0,1,2,\cdots$$

then $F(\eta,\mu,\beta;x,t)$ is always positive for all $x>0, t\in(0,x)$, since that $\left(1-\frac{t^\beta}{x^\beta}q^\lambda\right)>0,$ for $\mathrm{Re}(\beta)>0,\ 0<q<1, \mathrm{Re}(\lambda)>0,$ and the other terms are also positive under the conditions established in the theorem.

Multiplying (38) by $F(\eta,\mu,\beta;x,t),$ taking the q-integration from the result with respect to t from 0 to x, and keeping in mind the definition of $I_q^{\eta,\mu,\beta}(\cdot)$ operator, we get

$$I_q^{\eta,\mu,\beta}(f(x)g(x)p(x)\ +p(y)I_q^{\eta,\mu,\beta}(f(x)g(x)\ +f(y)g(y)I_q^{\eta,\mu,\beta}(p(x)\ +$$

$$\beta[1/\beta]_q\frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)}f(y)g(y)p(y)\geq g(y)I_q^{\eta,\mu,\beta}(f(x)p(x)\ +$$

$$g(y)p(y)I_q^{\eta,\mu,\beta}(f(x)\ +f(y)I_q^{\eta,\mu,\beta}(g(x)p(x)\ +f(y)p(y)I_q^{\eta,\mu,\beta}(g(x)), \tag{40}$$

where we use (32).

Similarly, multiplying (40) by $F(\varepsilon,\nu,\alpha;x,y),$ taking the q-integration from the result with respect to y from 0 to x, and keeping in mind the definition of $I_q^{\varepsilon,\nu,\alpha}(\cdot)$ operator, we get (37).

If in the Theorem 4 we put $\varepsilon=\eta,\ \nu=\mu,\ \alpha=\beta$ we get:

Corollary 1. Let p be a positive function on $[0,\infty)$ f,g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\beta[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\eta,\mu,\beta}\left(f(x)g(x)p(x)\right) \geq I_q^{\eta,\mu,\beta}\left(g(x)\right)\ I_q^{\eta,\mu,\beta}\left(f(x)p(x)\right)+$$

$$I_q^{\eta,\mu,\beta}\left(g(x)p(x)\right)\ I_q^{\eta,\mu,\beta}\left(f(x)\right) - I_q^{\eta,\mu,\beta}\left(p(x)\right)\ I_q^{\eta,\mu,\beta}\left(f(x)g(x)\right), \tag{41}$$

where $x>0,\ 0<q<1,\ \mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0, \mathrm{Re}(\eta)>-1$.

If $p(x)=x^{\lambda}$ in the Theorem 4 we have:

Corollary 2. Let f and g be synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu,\beta}(\cdot)$ a fractional q-integral operator, as defined by (9), then

$$\alpha[1/\alpha]_q \frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)} I_q^{\eta,\mu,\beta}\left(x^{\lambda} f(x)g(x)\right) + \beta[1/\beta]_q \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu,\alpha}\left(x^{\lambda} f(x)g(x)\right) \geq$$

$$I_q^{\varepsilon,\nu,\alpha}\left(g(x)\right)\ I_q^{\eta,\mu,\beta}\left(x^{\lambda} f(x)\right) + I_q^{\varepsilon,\nu,\alpha}\left(x^{\lambda} g(x)\right)\ I_q^{\eta,\mu,\beta}\left(f(x)\right) + I_q^{\varepsilon,\nu,\alpha}\left(f(x)\right)\ I_q^{\eta,\mu,\beta}\left(x^{\lambda} g(x)\right)+$$

$$I_q^{\varepsilon,\nu,\alpha}\left(x^{\lambda} f(x)\right)\ I_q^{\eta,\mu,\beta}\left(g(x)\right) - \alpha[1/\alpha]_q \frac{\Gamma_q(\varepsilon+1+\lambda/\alpha)}{\Gamma_q(\nu+\varepsilon+1+\lambda/\alpha)} x^{\lambda}\ I_q^{\eta,\mu,\beta}\left(f(x)g(x)\right) -$$

$$\beta[1/\beta]_q \frac{\Gamma_q(\eta+1+\lambda/\beta)}{\Gamma_q(\mu+\eta+1+\lambda/\beta)} x^{\lambda}\ I_q^{\varepsilon,\nu,\alpha}\left(f(x)g(x)\right) \tag{42}$$

where $x>0,\ 0<q<1,\ \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\eta+1+\lambda/\beta)>0,$ $\mathrm{Re}(\varepsilon+1+\lambda/\alpha)>0, \mathrm{Re}(\eta)>-1, \mathrm{Re}(\varepsilon)>-1.$

Proof. This result is obtained directly from (37) by replacing $p(x)$ by x^{λ} and using (31).

Now, if we make $\alpha=\beta=1$ then the Theorem 4 reduces to the following q-integral inequality involving basic analogue of Kober fractional integral operator:

Corollary 3. Let p be a positive function on $[0,\infty)$, f, g synchronous functions on $[0,\infty)$ and $I_q^{\eta,\mu}(\cdot)$ a fractional q-integral operator, as defined by (5), then

$$\frac{\Gamma_q(\varepsilon+1)}{\Gamma_q(\nu+\varepsilon+1)} I_q^{\eta,\mu}\left(f(x)g(x)p(x)\right) + \frac{\Gamma_q(\eta+1)}{\Gamma_q(\mu+\eta+1)} I_q^{\varepsilon,\nu}\left(f(x)g(x)p(x)\right) \geq$$

$$I_q^{\varepsilon,\nu}\left(g(x)\right)\ I_q^{\eta,\mu}\left(f(x)p(x)\right) + I_q^{\varepsilon,\nu}\left(g(x)p(x)\right)\ I_q^{\eta,\mu}\left(f(x)\right)+$$

$$I_q^{\varepsilon,\nu}\left(f(x)\right)\ I_q^{\eta,\mu}\left(g(x)p(x)\right) + I_q^{\varepsilon,\nu}\left(f(x)p(x)\right)\ I_q^{\eta,\mu}\left(g(x)\right)-$$

$$I_q^{\varepsilon,\nu}\left(p(x)\right)\ I_q^{\eta,\mu}\left(f(x)g(x)\right) - I_q^{\varepsilon,\nu}\left(f(x)g(x)\right)\ I_q^{\eta,\mu}\left(p(x)\right) \tag{43}$$

where $x>0,\ 0<q<1, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0, \mathrm{Re}(\eta)>-1, \mathrm{Re}(\varepsilon)>-1$.

Special Cases: i) Taking $\lim\limits_{q\to 1^-}$ in Theorem 4 and Corollaries 1-3, and additionally making $\alpha=\beta=1$ in Theorem 4 and Corollaries 1-2, we obtain integral inequalities involving Erdélyi-Kober operators.

ii) For $\lambda=0, \eta=\varepsilon=0, \alpha=\beta=1$ in Corollary 2 and using (11) we get,

$$\frac{x^{\nu}}{\Gamma_q(\nu+1)} I_q^{\mu}\big(f(x)g(x)\big)+\frac{x^{\mu}}{\Gamma_q(\mu+1)} I_q^{\nu}\big(f(x)g(x)\big)\geq$$

$$I_q^{\nu}\big(g(x)\big)\, I_q^{\mu}\big(f(x)\big)+I_q^{\nu}\big(f(x)\big)\, I_q^{\mu}\big(g(x)\big) \tag{44}$$

with $x>0,\ 0<q<1, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0.$

This result corresponds to quantun version of the same given by Öğünmez and Özkan [24, p. 5, No. (3.11)].

Taking $\lim\limits_{q\to 1^-}$ in (44) we obtain the result given by Belarbi and Dahmani [19, p. 188, No. (16)].

iii) For $\eta=\varepsilon=0$ in Corollary 3 and applying (11) we have,

$$\frac{x^{\nu}}{\Gamma_q(\nu+1)} I_q^{\mu}\big(f(x)g(x)p(x)\big)+\frac{x^{\mu}}{\Gamma_q(\mu+1)} I_q^{\nu}\big(f(x)g(x)p(x)\big)\geq$$

$$I_q^{\nu}\big(g(x)\big)\, I_q^{\mu}\big(f(x)p(x)\big)+I_q^{\nu}\big(g(x)p(x)\big)\, I_q^{\mu}\big(f(x)\big)+$$

$$I_q^{\nu}\big(f(x)\big)\, I_q^{\mu}\big(g(x)p(x)\big)+I_q^{\nu}\big(f(x)p(x)\big)\, I_q^{\mu}\big(g(x)\big)-$$

$$I_q^{\nu}\big(p(x)\big)\, I_q^{\mu}\big(f(x)g(x)\big)-I_q^{\nu}\big(f(x)g(x)\big)\, I_q^{\mu}\big(p(x)\big) \tag{45}$$

$x>0,\ 0<q<1, \mathrm{Re}(\mu)>0, \mathrm{Re}(\nu)>0,$

which is a known result [27, p. 456, No. (3.2)].

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