Revista Tecnocientífica URU
Universidad Rafael Urdaneta
Facultad de Ingeniería
Nº 6 Enero - Junio 2014
Depósito legal: ppi 201402ZU4464
ISSN: 2343 - 6360

Sobre una generalización de la función hipergeométrica de Gauss

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Dedicated to Professor Shyam Kalla
On the occasion of his 76th birthday

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Recibido: 29-08-2013 Aceptado: 13-03-2014

Resumen

El trabajo trata una nueva generación de la función hipergeométrica de Gauss, ${}_rF^{\tau,\beta}(a,b;c;z)$. Se demuestran las propiedades básicas de esta función (representación en serie, formulas diferenciales, relaciones fraccionales, representación integral, relación de tipo Erdelyi).

Palabras y frases claves: r-función hipergeométrica, función gamma (τ,β), – función hipergeométrica confluente.

On one generalization of the Gauss' hypergeometric function

Abstract

The paper devoted to the new generalization of the hypergeometric Gauss' function, ${}_rF^{\tau,\beta}(a,b;c;z)$. The basic properties of this function (the representation by series, the differential formulas, the fractional relations, integral representations, the relation of type Erdelyi) are proved.

2000 Mathematical Classification: 33C05, 33C20, 33B15.

Key words and phrases: r - hypergeometric function, gamma-function, (τ,β), – confluent hypergeometric function .

Introduction

Further studying of the special functions is prospective and very useful for the different branches of science.

The continuous development of the mechanics of solid medium, mathematical physics, probability theory, aerodynamics, astronomy and others has led to the generalization and creation of new classes of special functions [1], [2], [3].

It this article we consider the r – generalized Gauss' hypergeometric function, its properties.

Main results

Let us consider the r – generalized Gauss' hypergeometric function in the following form:

$$ {}_rF^{\tau,\beta}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \times \tag{1}$$

$$ \times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt, $$

where $\operatorname{Re}c > \operatorname{Re}b > 0, \{\tau,\beta\} \subset R, \tau > 0, \tau-\beta < 1; r > 0; r = 0, |z| < 1; \operatorname{Re}\gamma > \operatorname{Re}\alpha > 0,$ is the classical beta-function [5], ${}_1\Phi_1^{\tau,\beta}(...)$ is the (τ,β) generalized confluent hypergeometric function [4]:

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{1}{B(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\ {}_1\Psi_1\left[\begin{matrix}(c;\tau)\\(c;\beta)\end{matrix}\middle| zt^{\tau}\right]dt, \tag{2}$$

where ${}_1\Psi_1(...)$ is the Fox-Wright function [1]. As $\tau = \beta = 1, \alpha = \gamma$ in (1) we have:

$$ {}_r\tilde{F}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ e^{-\frac{r}{t(1-t)}}dt. \tag{3}$$

As $\beta = \tau$, $r = 0$ in (1) we have the Gauss' hypergeometric function $F(a,b;c;z)$ [5].

Theorem 1 (on the representation of the function ${}_rF^{\tau,\beta}(a,b;c;z)$ by the series).

As the conditions:

$$ r \in \mathbf{C}, r > 0; r = 0, |z| < 1, z \in \mathbf{C}; \{a,b,c\} \subset \mathbf{C}, \operatorname{Re}c > \operatorname{Re}b > 0; \tag{4}$$
$$ \{\tau,\beta\} \subset R, \tau > 0, \tau-\beta < 1,\ \operatorname{Re}\gamma > \operatorname{Re}\alpha > 0 $$

the following formula for ${}_rF^{\tau,\beta}(a,b;c;z)$ *is valid:*

$$ {}_rF^{\tau,\beta}(a,b;c;z) = \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\ {}_{\tau,\beta}B_{\alpha}^{\gamma}(b+n,c-b;r)\frac{z^n}{n!}, \tag{5}$$

where $(a)_n$ *is the Pochhammer' symbol,* ${}_{\tau,\beta}B_{\alpha}^{\gamma}(...)$ *is the* (τ,β) *- generalized beta-function [4]:*

$$ {}_{\tau,\beta}B_{\alpha}^{\gamma}(x,y;r;\delta;\omega) = \int_0^1 t^{x-t}(1-t)^{y-1}\ {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t^{\delta}(1-t)^{\omega}}\right)dt, \tag{6}$$

here $\operatorname{Re}x > 0, \operatorname{Re}y > 0, \delta > 0, \omega > 0, {}_1\Phi_1^{\tau,\beta}(...)$ is the function (2). The series (5) converges absolutely as $|z| < 1$.

Proof. Using the function ${}_{\tau,\beta}B_{\alpha}^{\gamma}(x,y;r;\delta;\omega)$, its properties, the legality of interchanging the order of integration and summation, we have:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)\Gamma(a)}\sum_{n=0}^{\infty}\Gamma(a+n)\frac{z^n}{n!}\times $$

$$ \times\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt= $$

$$ =\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\;{}_{\tau,\beta}B_{\alpha}^{\gamma}\left(b+n,c-b;r\right)\frac{z^n}{n!} $$

Let us prove absolute convergence of series (5) as $|z|<1$. The series (5) is the power series:

$$ {}_rF^{\tau,\beta}(a,b;c;z)=A\sum_{n=0}^{\infty}c_n z^n, $$

where

$$ A=\frac{1}{B(b,c-b)\Gamma(a)},\qquad c_n=\frac{\Gamma(a+n)\,{}_{\tau,\beta}B_{\alpha}^{\gamma}(b+n,c-b;r)}{n!}. $$

Let us consider asymptotic behavior of c_n as $n\to\infty$. Using the formula for Γ – function [5]:

$$ \Gamma(z)=\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}[1+O(z^{-1})],z\to\infty, \tag{7} $$

we receive:

$$ {}_{\tau,\beta}B_{\alpha}^{\gamma}\left(\tilde{a}+n,\tilde{b};r\right)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\frac{\Gamma(\tilde{b}-k)\Gamma(\tilde{a}-k+n)(-r)^k}{\Gamma(\tilde{a}+\tilde{b}-2k+n)k!}= $$

$$ =\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\frac{\Gamma(\tilde{b})\Gamma(1-\tilde{b})\sqrt{2\pi}(\tilde{a}+n-k)^{\tilde{a}-k+n-\frac{1}{2}}}{(-1)^k\Gamma(1-\tilde{b}+k)\sqrt{2\pi}(\tilde{a}+\tilde{b}-2k+n)^{\tilde{a}+\tilde{b}-2k+n-\frac{1}{2}}}\times $$

$$ \times\frac{e^{-\tilde{a}+k-n}}{e^{-\tilde{a}-\tilde{b}+2k-n}}\frac{(-r)^k}{k!}=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k\tau)}{\Gamma(\gamma+k\beta)}\times $$

$$ \times\frac{1}{\Gamma(1-\tilde{b}+k)}\frac{(nr)^k}{k!}=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{b}}G_{1,3}^{1,1}\left[\tau n\middle|\begin{matrix}1-\alpha\\0,1-\gamma,\tilde{b}\end{matrix}\right], $$

where $G_{p,q}^{m,n}[...]$ is the G-Mejer' function [1].

By the help of formula [1]:

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\right.\right]=G_{p,q}^{m,n}\left[\frac{1}{z}\left|\begin{matrix}1-b_1,...,1-b_q\\ 1-a_1,...,1-a_p\end{matrix}\right.\right]$$

we have as $n\to\infty$.

$${}_{\tau,\beta}B_{\alpha}^{\gamma}\left(\tilde{a}+n,\tilde{b};r\right)=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}G_{1,3}^{1,1}\left[\frac{1}{\tau n}\left|\begin{matrix}1,\gamma,1-\tilde{b}\\ \alpha\end{matrix}\right.\right]=$$

$$=\frac{\Gamma(\gamma)\Gamma(\tilde{b})\Gamma(1-\tilde{b})}{\Gamma(\alpha)n^{\tilde{b}}}\frac{\Gamma(1-\gamma)\Gamma(\tilde{b})}{\Gamma(1-\alpha)}=\frac{\Gamma(\gamma)\Gamma^2(\tilde{b})\Gamma(1-\tilde{b})\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)n^{\tilde{b}}}=\frac{C}{n^{\tilde{b}}},$$

where

$$C=\frac{\Gamma(\gamma)\Gamma^2(\tilde{b})\Gamma(1-\tilde{b})\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(1-\alpha)}.$$

Consequently, as $\tilde{a}=b,\tilde{b}-c-b,n\to\infty,$ (7) we get:

$$c_n=\frac{\Gamma(a+n)C}{n!n^{c-b}}=\frac{C\sqrt{2\pi}(a+n)^{a+n-\frac{1}{2}}e^{-a-n}}{n!n^{c-b}}.$$

$$R=\lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|}=\lim_{n\to\infty}\left|\frac{c\sqrt{2\pi}(a+n+1)^{a+n+\frac{1}{2}}e^{-a-n-1}}{(n+1)!(n+1)c-b}\right|=1.$$

The series (5) converges absolutely as $|z|<1$.

Theorem 2. *Under the conditions of existence of* ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following formulas of differentiation are hold:*

$$\frac{d\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz}=\frac{ab}{c}\,{}_rF^{\tau,\beta}(a+1,b+1;c+1;z),\qquad(8)$$

$$\frac{d^n\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz^n}=\frac{(a)_n(b)_n}{(c)_n}\,{}_rF^{\tau,\beta}(a+n,b+n;c+n;z),\qquad(9)$$

$$z\frac{d\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz}=a\left({}_rF^{\tau,\beta}(a+1,b;c;z)-{}_rF^{\tau,\beta}(a,b;c;z)\right),\qquad(10)$$

$$\frac{d^n}{dz^n}\left(z^{a+n-1}\,{}_rF^{\tau,\beta}(a,b;c;z)\right)=(a)_nz^{a-1}\,{}_rF^{\tau,\beta}(a+n,b;c;z).\qquad(11)$$

Proof. Let us prove (9), (11). We have

$$\frac{d^n\,{}_rF^{\tau,\beta}(a,b;c;z)}{dz^n}=\frac{d^n}{dz^n}\left(\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\times\right.$$

$$\left.\times{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt\right)=\frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}(1-zt)^{-a}\times$$

$$\times{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)(-a)(-a-1)...(-a-n+1)(-1)^n dt=$$

$$=\frac{(a)_n(b)_n}{(c)_n}{}_rF^{\tau,\beta}(a+n,b+;c+n;z);$$

$$\frac{d^n}{dz^n}\left(z^{a+n+1}{}_rF^{\tau,\beta}(a,b;c;z)\right)=\frac{d^n}{dz^n}\left(\frac{1}{B(b,c-b)}\sum_{k=0}^{\infty}\Gamma(a+k)\frac{z^{a+n+k-1}}{k!}\times\right.$$

$$\times{}_{\tau,\beta}B_\alpha^\gamma(b+k,c-b;r)=\frac{1}{B(b,c-b)\Gamma(a)}\sum_{k=0}^{\infty}\Gamma(a+k)\,{}_{\tau,\beta}B_\alpha^\gamma(b+k,c-b;r)\times$$

$$\times\frac{z^{a+k-1}}{k!}(a+n+k-1)(a+n+k-2)...(a+k)=$$

$$=(a)_n z^{a-1}{}_rF^{\tau,\beta}(a+n,b;c;z).$$

Theorem 3. *For* $z,r\in\mathbf{C},r>0;r=0,|z|<1;\{\tau,\beta\}\subset R_+,\ \tau-\beta<1,\ \{a,b,c\}\subset\mathbf{C}$, Re$\gamma>$ Re$\alpha>0$,Re$c>$ Re$b>0$ *the following functional relations are valid:*

$${}_rF^{\tau,\beta}(a+1,b;c;z)-{}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=\frac{b}{c}z\,{}_rF^{\tau,\beta}(a+1,b+1;c+1;z),\tag{12}$$

$$b\ {}_rF^{\tau,\beta}(a,b+1;c;z)+(c-b-1){}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=(c-1){}_rF^{\tau,\beta}(a,b;c-1;z).\tag{13}$$

Proof. Let us prove (13). Using (2):

$$b\ {}_rF^{\tau,\beta}(a,b+1;c;z)+(c-b-1){}_rF^{\tau,\beta}(a,b;c;z)=$$

$$=\frac{1}{B(b+1,c-b-1)}\int_0^1 t^b(1-t)^{c-b-2}(1-zt)^{-a}{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt+$$

$$=\frac{c-b-1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt=$$

$$=\frac{c}{B(b,c-b-1)}\int t^{b-1}(1-t)^{c-b-2}(1-zt)^{-a}{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt=$$

$$=(c-1){}_rF^{\tau,\beta}(a,b;c-1;z).$$

Theorem 4. *In the case of fulfilling of the conditions of existence of* ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following integral representations for* ${}_rF^{\tau,\beta}(a,b;c;z)$ *are valid*:

$${}_rF^{\tau,\beta}(a,b;c;z)=\frac{2}{B(b,c-b)}\int_0^{\infty}\frac{(ch\omega)^{2a-2c+1}(sh\omega)^{2b-1}}{(ch^2\omega-zsh^2\omega)^a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{rch^4\omega}{sh^2\omega}\right)d\omega, \tag{14}$$

$${}_rF^{\tau,\beta}(a,b;c;z)=\frac{2^{b-a}}{B(b,c-b)}\int_0^\infty\frac{(ch\theta)^{2a-2c+1}(sh\theta)^{2c-a-b-1}}{\left(\frac{1}{2}-z+\frac{1}{2}ch\theta\right)^a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{rsh^4\theta}{2(ch\theta-1)^3}\right)d\theta, \tag{15}$$

$${}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^\infty e^{-bt}(1-e^{-t})^{c-b-1}(1-ze^{-t})^{-a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{re^t}{1-e^{-t}}\right)dt. \tag{16}$$

Proof of these relations can be straightforward make using in (1) the such changes of variable respectively:

$$t=\frac{sh^2\omega}{ch^2\omega};\qquad t=\frac{2}{ch\theta+1};\qquad t=e^{-t}.$$

Theorem 5. (*The relation of type Erdelyi for* ${}_rF^{\tau,\beta}(a,b;c;z)$)

As the conditions (4) for ${}_rF^{\tau,\beta}(a,b;c;z)$ *the following relation is hold*:

$${}_rF^{\tau,\beta}(a,b;c;z)=(1-z)^{-a}\,{}_rF^{\tau,\beta}\left(a,c-b;c;\frac{z}{z-1}\right). \tag{17}$$

Proof. Using (1) and the substitution

$t=1-v$

we receive:

$${}_rF^{\tau,\beta}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^\infty(1-v)^{b-1}v^{c-b-1}(1-\frac{z}{z-1}v)^{-a}\times$$

$$\times(1-z)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{(1-v)v}\right)dv=$$

$$=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1(1-v)^{b-1}v^{c-b-1}(1-\frac{z}{z-1}v)^{-a}\times$$

$$\times {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{(1-v)v}\right)dv=(1-z)^{-a}\quad {}_rF^{\tau,\beta}\left(a,c-b;c;\frac{z}{z-1}\right).$$

Here: $1-z(1-v)=(1-z)\left(1-\frac{z}{z-1}v\right).$

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